Of Bhāskara's treatise on algebra there is one complete manuscript:

Manuscript of the *Bija Ganita* on paper, $4\frac{3}{4}$ by 13 in., Sanskrit, 42 leaves (84 pages), copied at Benares by Nundarama in 1825.

There is also in the library a copy of the first printed edition.[1]

Altogether there are fifteen manuscripts of the works of Bhāskara in the library, and the first Sanskrit printed editions of the above-mentioned treatises.

Of the best-known algebras of the Moslem civilization there are two rare specimens. The first is the *'Ilm al-jabr w'al muqabalah* (*Science of reduction and cancellation*) of Mohammed ibn Mûsâ al-Khowârizmî (*c.* 830), an Arabic manuscript on paper, $6\frac{1}{2}$ by $9\frac{1}{4}$ in., 132 numbered pages, without date, but probably *c.* 1800. Suter does not list any Arabic manuscripts in Europe, but the Rosen translation (London, 1831) was made from a copy in the Bodleian Library at Oxford, which was the only one that the translator had been able to find. There must, however, be others in the European collections of Orientalia.

The second noteworthy Oriental manuscript on algebra is one of the treatise of Omar Khayyam ('Omar ibn Ibrâhîm al-Khayyâmî, Giyât ed-dîn, Abû'l-Fath, *c.* 1044–1123/24). This is an Arabic copy on paper, $6\frac{3}{4}$ by $9\frac{1}{2}$ in., 49 numbered leaves (98 pages), without date, but probably written *c.* 1800. The text is identical, at least in its main features, with the Arabic version of Woepcke, Paris, 1851. Woepcke examined three manuscripts: (1) one in the Bibliothèque Nationale; (2) a fragment, also in the same library; and (3) one in Leyden. Suter mentions also one in the India Office, London. The text is written with unusual elegance and distinctness.

While not bearing upon the mathematical question, but upon the work of Omar Khayyam, it may be of interest to oriental scholars to know that there is in the library a beautifully written manuscript of his *Rubaiyat*, 78 numbered pages, $3\frac{1}{2}$ by $5\frac{3}{4}$ in., without date but probably the sixteenth or seventeenth century, in Persian stamped binding, gilded leather.

Of the oriental manuscripts on geometry in the library, two are of special interest. The first is the Arabic translation of Books I–VI of Euclid's *Elements*, made by Ishâq ibn Honein, *c.* 890, and revised by Tâbit ibn Qorra at about the same time. The manuscript is on paper, $6\frac{1}{2}$ by $11\frac{1}{4}$ in., 95 numbered leaves (190 pages). It is dated 751 A.H. (1350/51 A.D.).[2]

The second of the interesting manuscripts of Euclid is the Chinese version of Matteo Ricci (Li Ma-do, in Chinese), translated in 1603–1607 with the help of two learned mandarins. This copy is on paper, 6 by 11 in., and is bound in eight parts in two folders. It is a fine piece of Chinese writing and is undated, but probably was copied *c.* 1800. There is also in the library the first printed edition of this version.

The great Persian writer Nasîr ed-dîn al-Tûsî (1201–1274) left many works of importance, some of which are represented by manuscripts in this library, as follows:

[1] Fac-simile in *History*, II, 426.

[2] Fac-simile in *History*, I, 173.

1. *Kitab-i sî fasl* (*Book of thirty sections*), manuscript on paper, 4½ by 8 in., Persian, 32 leaves (64 pages, 2 blank), copied in 1354 A.D.

2. The *Centiloquium of Ptolemy*, manuscript on paper, 6 by 10 in., Persian, 27 leaves (54 pages). Other manuscript copies exist in Leyden, Florence, Constantinople, Cairo, and the British Museum.[1]

3. *Risâle-i; bist bâb*, on the astrolabe, manuscript on paper, 4½ by 7⅛ in., Persian, 23 leaves (46 pages). Other manuscript copies exist in Petrograd, Florence, Oxford, Constantinople, and the British Museum.

'Abderrahmân al-Khâzinî, Abû Mansûr, known as Abû'l Fath of Bagdad, wrote *c.* 1136 A.D. the Sinjaric Tables, dedicated to the sultan Sinjar, of which there is a copy in the Vatican. What is probably a complete copy of this work is in this library, dated 1011 A.H. (1602/3 A.D.), on paper, Persian manuscript, 86 leaves (172 pages), 6⅛ by 9¼ in.

As a piece of writing, the most interesting work in the collection of Orientalia is a Persian manuscript of the tables of Ulugh Beg (1393–1449), the prince astronomer of Samarkand. It consists of 194 numbered leaves (388 pages, 4 blank + 1 with notes, 6 by 9⅝ in. It bears the date 1214 A.H. (1836 A.D.)). While not fully checked, it contains the catalogue of stars and apparently most if not all of the other tables as given by Sédillot and quoted by Knobel.

Of the commentators on Ulugh Beg the most prominent was 'Abdel 'alî ibn Mohammed ibn al-Hosein, al Barjendî, who died *c.* 930 A.H. (*c.* 1524 A.D.). There is in the library a copy of this work, manuscript on paper, 5½ by 9¼ in., Persian, 312 leaves (624 pages), written in 965 A.H. (1558 A.D.).

A second copy of this same manuscript by al Barjendî is in the library. Manuscript on paper, 5¾ by 9 in., 258 leaves (516 pages), not dated, but *c.* 1550.

There is also a manuscript of a later commentator, Molla Ali Qoshchy (?), paper, 4½ by 8½ in., Persian, 52 leaves (104 pages), apparently *c.* 1750.

The last of the notable scholars of Islam was Behâ ed-din al-'Âmilî (1547–1622), probably a Persian. In the library are two manuscripts relating to his works, as follows:

1. *Kholâsat al-Hisâb* (*Essence of Arithmetic*), manuscript on paper, 7¾ by 11⅞ in., Persian, 320 leaves (640 pages), together with a commentary of 58 leaves (116 pages), in a different hand. The *Kholâsat* was written *c.* 1600. A date at the end of the commentary is 1024 A.H. (1615 A.D.), which may refer to the original treatise. The manuscript of the commentary bears no date. The manuscript of the text proper was written in 1135 A.H. (1723 A.D.). There are manuscripts in Berlin, Munich, British Museum, India Office, Cairo, Constantinople, and Paris.

2. Commentary on the above, manuscript on paper, 6½ by 9½ in., Persian, 30 leaves (60 pages), no date, but probably 17th century. The commentary is found in some of the manuscripts referred to above.

There are in the library a large number of manuscripts on astrology from

[1] On all such statements see Suter, *Abhandlungen*, Heft X, Leipzig, 1900.

India and adjacent regions. These are semi-mathematical in that they make more or less use of astronomical tables. Of these only two will be mentioned.

The first is the *Dyevagna-Kamadenuva*, the greatest of the Ceylon treatises on the subject. This copy is modern, on 190 palm leaves (380 pages), $1\frac{3}{4}$ by $12\frac{1}{4}$ in.

The second is a Madras copy of the *Vākyakaranaya*, on 117 palm leaves (234 pages), $2\frac{1}{4}$ by $6\frac{3}{4}$ in. The copy is modern.

Of the Chinese classics, represented in the library by a large number of printed works, the most interesting manuscript is one of the *I-king*.[1] This is a silk roll $16\frac{3}{4}$ in. by 6 ft. 4 in. The original is of uncertain date, but was probably written by Wön-wang (1182–1135 B.C.), and perhaps was based upon the works of earlier writers. This copy is modern and is a beautiful piece of Chinese writing in black and red.

The oldest manuscripts in the library are, naturally, the Babylonian cylinders. There are several bearing upon arithmetic, including receipts, tax lists, and bills. The most interesting one, however, is a cylinder $2\frac{1}{4}$ by $3\frac{1}{4}$ in., found at Warka, the Biblical Erech (Genesis, X, 10). It is a school boy's exercise tablet for practice in making numerals, and is filled with these forms. It comes from a mass of cylinders all of the Hammurabi period, *c.* 2100 B.C.

Concerning the Japanese manuscripts in the library it is impossible, in the space allowable, to speak in any great detail. There are more than a hundred specimens, including the complete *Kiku genpō chō ken* (on surveying) by Nobutomo Ogino (1718); the *Chō ken Tō monki* (also on surveying) by Murai Masahiro (*c.* 1732);[2] the *Honbun Hassen Hio* (on trigonometric tables) by Miju Rakŭsai (*c.* 1815); the *Saku yen riu kwai gi* (on the *yenri* theory applied to mensuration) by Toshihisa Isawaki (1831); the *Sampō Tengen jutsu* (algebra) by Moriyoshi Hanai; the *Yendan jutsu Kwaisho* (algebra) of Sataka Fusataka Iwasa and Shigeyuki Kato (*c.* 1865); the *Ken jutsu Denpō* (on surveying) of Hiroyoshi Ogawa (*c.* 1830); one of the well-known Hundred-Question collections copied by Nobujiro Hirata (*c.* 1818); the *Iyen Sampō* (on circles), written in 1825 by the celebrated Wada Yenzō Nei (1787–1840) and probably copied by one of his pupils at about the same time; a manuscript of the eighteenth century on Seki Kōwa's *Fukudai* process, in which Seki makes a step toward the theory of determinants; an extensive treatise in twelve volumes, the *Ichigen Kappo* by Hoshuku Iriye (1760); and a large number of treatises of the ancient *Wasan* (native mathematics) by minor or unnamed writers of the eighteenth and early nineteenth centuries. A complete list of this material, made by a Japanese scholar, would be of importance.

[1] *History*, I, 25.

[2] Fac-similes in *History*, II, 358, 359, 614.

ON A CENTRAL DIFFERENCE SUMMATION FORMULA.[1]

By W. A. JENKINS, Harvard University.

1. Statement of the Problem. It is often desirable in applied mathematics, when a number of equidistant statistical ordinates are known, to find approximately the sum of the given ordinates along with a certain number of interpolated values, in terms of the given ordinates and their differences. The solution of this problem may be applied equally well to the summation of the ordinates of any continuous function when it is impracticable to compute all the values desired in the sum, and also to obtaining the approximate definite integral. To this problem three solutions have been offered, the relative merits of which I propose to discuss.

2. Solutions. Let the known ordinates be $u_0, u_1, \cdots, u_m$, corresponding to the values of the argument $0, 1, 2, \cdots, m$. Let each interval be divided into n equal parts and an ordinate be interpolated between each. This problem was first solved by Sir J. W. Lubbock[2] in 1829, in terms of the advancing differences of u_0 and u_m as follows:

$$\sum_{x=0}^{mn-1} u_{x/n} = n\sum_{x=0}^{m-1} u_x + \frac{n-1}{2}(u_m-u_0) + \frac{n^2-1}{12n}(\Delta u_0-\Delta u_m) + \frac{n^2-1}{24n}(\Delta^2 u_m-\Delta^2 u_0)$$
$$+ \frac{(n^2-1)(19n^2-1)}{720n^3}(\Delta^3 u_0-\Delta^3 u_m) + \frac{(n^2-1)(9n^2-1)}{480n^3}(\Delta^4 u_m-\Delta^4 u_0) \qquad \text{(I)}$$
$$+ \frac{(n^2-1)(863n^4-145n^2+2)}{60480n^5}(\Delta^5 u_0-\Delta^5 u_m)+\cdots.$$

Augustus De Morgan[3] modified this solution in 1842 to involve advancing differences of u_0 and retiring differences of u_m as follows:

$$\sum_{x=0}^{mn-1} u_{x/n} = n\sum_{x=0}^{m-1} u_x + \frac{n-1}{2}(u_m-u_0) + \frac{n^2-1}{12n}(\Delta u_0-\Delta u_{m-1}) - \frac{n^2-1}{24n}(\Delta^2 u_0+\Delta^2 u_{m-2})$$
$$+ \frac{(n^2-1)(19n^2-1)}{720n^3}(\Delta^3 u_0-\Delta^3 u_{m-3}) - \frac{(n^2-1)(9n^2-1)}{480n^3}(\Delta^4 u_0+\Delta^4 u_{m-4}) \qquad \text{(II)}$$
$$+ \frac{(n^2-1)(863n^4-145n^2+2)}{60480n^5}(\Delta^5 u_0-\Delta^5 u_{m-5})+\cdots.$$

W. S. B. Woolhouse,[4] in a paper in 1864, developed formulæ I and II, and in the development derived the following central difference formula, III, which seems to be much more convergent than either of the other two forms. He

[1] Presented to the Michigan Section of the Association, April 2, 1925.
[2] J. W. Lubbock, *Trans. Camb. Phil. Soc.*, III, 324.
[3] Augustus De Morgan, *Diff. Int. Calc.*, 318.
[4] W. S. B. Woolhouse, *Jour. Inst. Actuaries*, XI, 307.

does not seem to have realized the advantage of this but passed immediately to formula II. Nor does this central difference formula seem to have been mentioned in writings until 1924, when it was developed by a Danish mathematician, Steffensen, in a somewhat modified form. In a form parallel to I and II, it is

$$\sum_{x=0}^{mn-1} u_{x/n} = n\sum_{x=0}^{m-1} u_x + \frac{n-1}{2}(u_m - u_0) + \frac{n^2-1}{24n}(\Delta u_0 + \Delta u_{-1} - \Delta u_m - \Delta u_{m-1})$$
$$+ \frac{(n^2-1)(11n^2+1)}{1440n^3}(\Delta^3 u_{m-1} + \Delta^3 u_{m-2} - \Delta^3 u_{-1} - \Delta^3 u_{-2}) \qquad \text{(III)}$$
$$+ \frac{(n^2-1)(191n^4+23n^2+2)}{120960n^5}(\Delta^5 u_{-2} + \Delta^5 u_{-3} - \Delta^5 u_{m-2} - \Delta^5 u_{m-3}) + \cdots.$$

If, in I, II, and III, we multiply through by ω/n, where ω is the number of units argument in the interval between u_0 and u_1, and let n become infinite, we have

$$\int_0^m u_x dx = \omega\left[\sum_{x=0}^{m-1} u_x + \frac{1}{2}(u_m - u_0) + \frac{1}{12}(\Delta u_0 - \Delta u_m)\right.$$
$$+ \frac{1}{24}(\Delta^2 u_m - \Delta^2 u_0) + \frac{19}{720}(\Delta^3 u_0 - \Delta^3 u_m) \qquad \text{(Ia)}$$
$$\left. + \frac{3}{160}(\Delta^4 u_m - \Delta^4 u_0) + \frac{863}{60480}(\Delta^5 u_0 - \Delta^5 u_m) + \cdots\right].$$

$$\int_0^m u_x dx = \omega\left[\sum_{x=0}^{m-1} u_x + \frac{1}{2}(u_m - u_0) + \frac{1}{12}(\Delta u_0 - \Delta u_{m-1})\right.$$
$$- \frac{1}{24}(\Delta^2 u_0 + \Delta^2 u_{m-2}) + \frac{19}{720}(\Delta^3 u_0 - \Delta^3 u_{m-3}) \qquad \text{(IIa)}$$
$$\left. - \frac{3}{160}(\Delta^4 u_0 + \Delta^4 u_{m-4}) + \frac{863}{60480}(\Delta^5 u_0 - \Delta^5 u_{m-5}) + \cdots\right].$$

$$\int_0^m u_x dx = \omega\left[\sum_{x=0}^{m-1} u_x + \frac{1}{2}(u_m - u_0) + \frac{1}{24}(\Delta u_0 + \Delta u_{-1} - \Delta u_m - \Delta u_{m-1})\right.$$
$$+ \frac{11}{1440}(\Delta^3 u_{m-1} + \Delta^3 u_{m-2} - \Delta^3 u_{-1} - \Delta^3 u_{-2}) \qquad \text{(IIIa)}^1$$
$$\left. + \frac{191}{120960}(\Delta^5 u_{-2} + \Delta^5 u_{-3} - \Delta^5 u_{m-2} - \Delta^5 u_{m-3}) + \cdots\right].$$

3. Comparison of Formulæ. The variations between the three formulæ, I, II, and III, may be traced to the method of interpolation used. The first two are based upon Newton's formula

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{2!}\Delta^2 u_0 + \cdots,$$

[1] See H. L. Rice, *Theory and Practice of Interpolation*, 153–4.

the third upon any central difference formula; let us take Everett's formula

$$u_x = xu_1 + \frac{x(x^2-1)}{3!}\Delta^2 u_0 + \frac{x(x^2-1)(x^2-4)}{5!}\Delta^4 u_{-1} + \cdots$$
$$+ \xi u_0 + \frac{\xi(\xi^2-1)}{3!}\Delta^2 u_{-1} + \frac{\xi(\xi^2-1)(\xi^2-4)}{5!}\Delta^4 u_{-2} + \cdots,$$

where

$$\xi = 1 - x.$$

As to accuracy, which of course depends upon relative convergence, formula III is superior to I and II. This is due to the fact that central difference interpolation takes into account ordinates equally on both sides of the interpolated value, whereas Newton's formula does not. We shall examine and approximately determine the error involved in each formula later. Formula II, however, has the advantage of involving only the differences of the ordinates u_0 to u_m, while I and III necessitate the obtaining of further values of the function; formula I values beyond u_m; formula III values beyond u_m and before u_0; formula III involving the same number of additional ordinates as formula I. Calculations by means of formula III are shorter as fewer terms are involved than in either I or II.

4. Development of Formulæ. The development of I and II are well known.[1] Formula III may be developed in several ways, perhaps the most simple and direct of them being from Everett's formula.

Let

$$(a)_n = \frac{a!}{n!(a-n)!}.$$

Interpolating,

$$u_0 = u_0,$$
$$u_{1/n} = \left(\frac{1}{n}\right)u_1 + \left(\frac{1}{n}+1\right)_3\Delta^2 u_0 + \left(\frac{1}{n}+2\right)_5\Delta^4 u_{-1} + \cdots.$$
$$+ \left(1-\frac{1}{n}\right)u_0 + \left(1-\frac{1}{n}+1\right)_3\Delta^2 u_{-1} + \left(1-\frac{1}{n}+2\right)_5\Delta^4 u_{-2} + \cdots,$$
. . . . . .
$$u_{1-(1/n)} = \left(1-\frac{1}{n}\right)u_1 + \left(1-\frac{1}{n}+1\right)_3\Delta^2 u_0 + \left(1-\frac{1}{n}+2\right)_5\Delta^4 u_{-1} + \cdots$$
$$+ \left(\frac{1}{n}\right)u_0 + \left(\frac{1}{n}+1\right)_3\Delta^2 u_{-1} + \left(\frac{1}{n}+2\right)_5\Delta^4 u_{-2} + \cdots.$$

Summing,

$$\sum_{x=0}^{n-1} u_{x/n} = u_0 + a(u_0+u_1) + b(\Delta^2 u_0 + \Delta^2 u_{-1}) + c(\Delta^4 u_{-1} + \Delta^4 u_{-2}) + \cdots,$$

[1] T. B. Sprague, *Jour. Inst. Actuaries*, XVIII, 305; J. W. Lubbock, *Jour. Inst. Actuaries*, V, 77; G. King, *Inst. Actuaries' Textbook*, 467; and above references.

where

$$a = \frac{n-1}{2}, \qquad b = -\frac{n^2-1}{24n}, \qquad c = \frac{(n^2-1)(11n^2+1)}{1440n^3}.$$

Interpolating further and summing,

$$\sum_{x=0}^{n-1} u_{x/n} = u_0 + a(u_0 + u_1) + b(\Delta^2 u_0 + \Delta^2 u_{-1}) + \cdots,$$
$$\sum_{x=n}^{2n-1} u_{x/n} = u_1 + a(u_1 + u_2) + b(\Delta^2 u_1 + \Delta^2 u_0) + \cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sum_{x=(m-1)n}^{mn-1} u_{x/n} = u_{m-1} + a(u_{m-1} + u_m) + b(\Delta^2 u_{m-1} + \Delta^2 u_{m-2}) + \cdots.$$

Whence, summing,

$$\sum_{x=0}^{mn-1} u_{x/n} = n\sum_{x=0}^{m-1} u_x + \frac{n-1}{2}(u_m - u_0) + \frac{n^2-1}{24n}(\Delta u_0 + \Delta u_{-1} - \Delta u_m - \Delta u_{m-1})$$
$$+ \frac{(n^2-1)(11n^2+1)}{1440n^3}(\Delta^3 u_{m-1} + \Delta^3 u_{m-2} - \Delta^3 u_{-1} - \Delta^3 u_{-2}) + \cdots,$$

which is formula III.

We may also start with formula I or II (commonly called Lubbock's formulæ) and by means of the relation

$$\Delta^k u_{a+1} = \Delta^k u_a + \Delta^{k+1} u_a$$

arrive at formula III. Instead of Everett's, we may base the interpolation upon Stirling's or Bessel's central difference interpolation formulæ.

5. Relative Accuracy of Formulæ. The proof of the superior convergence of formula III we may base upon Lagrange's interpolation formula with a remainder term,[1] which passes an $(n-1)$st degree curve through n given ordinates, $f(x_1)$, $f(x_2)$, $\cdots$ $f(x_n)$, and yields the interpolated $f(x)$. This is equivalent to using $(n-1)$st differences.

$$f(x) = \frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} f(x_1)$$
$$+ \cdots$$
$$+ \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} f(x_n) + R,$$

where

$$R = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!} f^{(n)}(\beta),$$

β being a value intermediate between x_1 and x_n. R is evidently the error in-

[1] Whittaker and Robinson, *Calculus of Observations*, 32.

volved in interpolation with n ordinates or $(n-1)$st differences. The only restriction here is that the function possess an nth derivative throughout the interval.

The respective errors in interpolation with Newton's and Everett's formulæ to $(2s-1)$st differences are

$$(x)_{2s}f^{(2s)}(\xi), \qquad (x+s-1)_{2s}f^{(2s)}(\eta),$$

where ξ is a value intermediate between u_0 and u_{2s}, and η is intermediate between u_{-s+1} and u_s. The summed errors between u_0 and u_m are, therefore,

$$\epsilon_1 = [f^{(2s)}(\xi) + f^{(2s)}(\gamma) + \cdots + f^{(2s)}(\rho)]\sum_{x=0}^{n-1}\binom{x}{n}_{2s},$$

$$\epsilon_2 = [f^{(2s)}(\eta) + f^{(2s)}(\delta) + \cdots + f^{(2s)}(\sigma)]\sum_{x=0}^{n-1}\binom{x}{n}+s-1\Big)_{2s},$$

where (ξ) lies between u_0 and u_{2s}; (η) lies between u_{-s+1} and u_s;

and (γ) lies between u_1 and u_{2s+1}; (δ) lies between u_{-s+2} and u_{s+1};

. . .

and (ρ) lies between u_m and u_{m+2s}; (σ) lies between u_{m-s+1} and u_{m+s}.

In general each derivative in ϵ_1 is approximately the same as the corresponding derivative in ϵ_2, so that we may consider the sums of the derivatives in each error as approximately the same. The value of the coefficient, however, is always smaller in the second case (formula III) than in the first (formula I) $(n = k > 1)$. Therefore we may in general expect a smaller error and more accurate results from formula III than from I. As to formula II, it is evident that in general the convergence is not as rapid even as in formula I.

6. Approximate Errors. We may derive an approximate value of the above errors as follows. Since ω, the number of units argument between u_0 and u_1, is one,

$f^{(2s)}(\xi)$ is approximately the same as [1] $\Delta^{2s}u_0$,

$f^{(2s)}(\gamma)$ is approximately the same as $\Delta^{2s}u_1$,

. . .

$f^{(2s)}(\rho)$ is approximately the same as $\Delta^{2s}u_{m-1}$,

and $f^{(2s)}(\eta)$ is approximately the same as $\frac{1}{2}(\Delta^{2s}u_{-s+1} + \Delta^{2s}u_{-s})$,

$f^{(2s)}(\delta)$ is approximately the same as $\frac{1}{2}(\Delta^{2s}u_{-s+2} + \Delta^{2s}u_{-s+1})$,

. . .

$f^{(2s)}(\sigma)$ is approximately the same as $\frac{1}{2}(\Delta^{2s}u_{m-s+1} + \Delta^{2s}u_{m-s})$.

[1] H. L. Rice, *Theory and Practice of Interpolation*, 102, 116.

The approximate error involved in using each formula to differences of order $(2s-2)$ becomes:

formula I, $T_{\mathrm{I}} = (\Delta^{2s-1}u_m - \Delta^{2s-1}u_0)\sum_{x=0}^{n-1}\binom{x}{n}_{2s}$,

formula III, $T_{\mathrm{III}} = (\Delta^{2s-1}u_{m-s} + \Delta^{2s-1}u_{m-s+1} - \Delta^{2s-1}u_{-s+1} - \Delta^{2s-1}u_{-s})\sum_{x=0}^{n-1}\binom{x}{n}+s\Big)_{2s+1}$.

T_{I} and T_{III} are evidently the values of the first terms neglected in the use of formulæ I and III respectively to differences of order $(2s-2)$.

These facts lead us to the conclusion that an approximate measure of the errors involved in formulæ I and III is the absolute magnitude of the first term neglected in the use of each formula; the formula with the smaller error being the more accurate and vice versa. This measure we could extend to apply to formula II in like manner. We find that this test succeeds in a majority of cases.

7. Conclusions. The accuracy of any of the three formulæ depends upon the rapidity with which the differences diminish with advancing order. In general formula III yields the more accurate results when used to the same order of differences as formulæ I and II. Using the three formulæ to $2k$th or $(2k-1)$st differences, we should expect greater accuracy from formula III if the $(2k+1)$st differences are approximately constant or are increasing regularly at the end of the summation at which their absolute magnitude is greater. Using the three formulæ to $2k$th or $(2k-1)$st differences, we should be in doubt as to which would yield the more accurate results if the $(2k+1)$st differences are diminishing regularly or are varying irregularly at the end of the summation at which their absolute magnitude is greater, but we can test approximately the error involved in each formula by computing the first neglected term; then the formula with the smaller neglected term is the more accurate.

8. Example. Let it be required to find $\sum_{x=30}^{49}(1+i)^{-x}l_x$, where i is the rate of interest, .03, and l_x is the number living at age x by the United States Life Tables No. 9. From values of the function differing in argument by five units, using the three formulæ to 4th differences, we obtain

Formula.	Value.	Neglected Term.	Error.
I.....	438,934.76	7.97	13.82
II.....	438,975.61	16.52	27.03
III.....	438,951.31	2.30	2.73

The neglected term correctly points out formula III, the central difference form, as the more accurate.

THE PATH OF LIGHT IN A GRAVITATIONAL FIELD.

By C. C. WYLIE, University of Iowa.

1. Introduction. The earliest work on the deflection of light in the gravitational field of the sun, so far as is known, is that of Von Soldner of Munich.[1] On the assumptions that light has weight, and that it is deflected according to Newton's law of gravitation, he computed the bending of a ray passing the limb of the sun. A factor 2 was erroneously included, making his result double the correct.

In 1911, Einstein,[2] using a very different method, but the same basic assumptions, computed the gravitational deflection of light passing the limb of the sun, and arrived at the same result (Von Soldner's, with the mistake corrected). Using modern values for the astronomical constants, this result, ordinarily termed the Newtonian deflection, is 0".87.

The relativity value, 1".75, was published by Einstein[3] in 1916. This is the value computed on the assumption that light is deflected according to the general theory of relativity.

The published discussions of the deflection of light are nearly all in writings on relativity. Since the derivation of the Newtonian deflection is a problem of celestial mechanics, it is not given, and as a consequence persons unfamiliar with the subject occasionally have difficulty.[4]

In the formula for the relativity deflection the mass of the sun appears and is considered a "length" of 1.47 kilometers. This conception of mass as a length is puzzling to many. A little search brought to light a reference in a scientific journal to mass as "three dimensional." The evaluation of the mass m gives the viewpoint of the relativity writers. Further, the reader can convince himself that it is a perfectly definite length, whether the units be the day and the mean distance of the earth from the sun, or the centimeter-second system.[5]

In the discussion of photographic plates taken at the time of a total eclipse the statement occasionally appears that according to the relativity theory the deflection should vary inversely as the radial distance from the center of the sun's image. This form is used for computing the theoretical deflection of each star from that at the limb. The proof is not given, so a simple one is included in this paper.

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[1] Bode, *Berliner Astronomisches Jahrbuch*, 1804. A reprint of a portion of his work was published by Lenard in *Annalen der Physik*, **65**, 593, 1921.

[2] *Annalen der Physik*, **35**, 898, 1911.

[3] *Annalen der Physik*, **49**, 769, 1916. Eclipse measurements indicate a deflection of about the amount here predicted: Mitchell, *Eclipses of the Sun*, 395–417; *Lick Observatory Bulletin*, No. 346.

[4] *Literary Digest*, No. 1803, Page 20, November 8, 1924. Report of address of Professor T. J. J. See, before the California Academy of Sciences. He received national advertising on the claim that he had found a mistake in the computation of the deflection which "any high school student" could understand, the omission of a factor 2.

[5] *Science*, **60**, 221, 1924. Very different values of m are derived with different sets of units.

It may be added that at the distance of the earth from the sun the deflection is practically that for an infinite distance.[1]

2. The Newtonian Deflection at the Limb of the Sun. The equations for the path, acceleration, and areal velocity-constant of an infinitesimal planet moving about the sun in accordance with the Newtonian law of gravitation are:[2]

$$r = p/(1 + e\cos\theta), \qquad \alpha = -k^2/pr^2, \qquad h = r^2 d\theta/dt.$$

Consider light just grazing the limb of the sun. Let us denote the radius of the sun by R, the acceleration of gravity at the surface of the sun by G, and the velocity of the light at this point by V_0. At this instant we have:

$$r = R, \qquad \theta = 0, \qquad d\theta/dt = V_0/R, \qquad \alpha = -G.$$

Substituting:

$$h = R^2V_0/R = RV_0, \qquad G = R^2V_0^2/pR^2, \qquad R = V_0^2/G(1 + e),$$
$$e = (V_0^2/RG) - 1, \qquad p = V_0^2/G, \qquad (1 + e) = V_0^2/RG.$$

Substitution shows e to be a number of seven digits, hence with sufficient accuracy we can write $e = V_0^2/RG$ and

$$1/e = RG/V_0^2. \tag{1}$$

As e is large, the path is a hyperbola. The asymptotes to the hyperbola make with the axis $\theta = 0$ the angle $\tan^{-1}\sqrt{e^2 - 1}$. With sufficient accuracy we can substitute e^2 for $(e^2 - 1)$, which gives for the value of the angle $\tan^{-1} e$. Half of the total deflection we desire is 90° minus this angle; that is, denoting the total deflection by D_0:

$$D_0/2 = \cot^{-1} e = \tan^{-1}(1/e) = \tan^{-1}(RG/V_0^2).$$

Since D is very small, we can express it in seconds of arc, as

$$D_0''/2 = RG/(V_0^2 \tan 1''),$$
$$D_0'' = 2RG/(V_0^2 \tan 1''). \tag{2}$$

An inspection of the derivation shows this formula will give the deflection at any distance from the center of the sun, if the distance is denoted by R, and the sun's attraction at that distance by G.

3. The Mass of the Sun a Length. From Newton's law, the acceleration at a distance r from the center of the sun is

$$\alpha = k^2M/r^2, \tag{3}$$

where k is a constant depending on the units adopted, and M is the mass of the

[1] H. S. Uhler, MONTHLY (*1922*, 47).
[2] Moulton, *Celestial Mechanics*, revised edition, page 93, and elsewhere.
Ibid., page 18, Problem 3. The A of this problem corresponds to the h of page 81.
Ibid., page 81.

sun. The mass m used by the writers on relativity is in units so chosen as to reduce the k^2 to unity. Further, the unit of time is the time in which light travels unit distance. Denoting the velocity of light by V, the unit of time is $1/V$ and the acceleration in this unit [1] is α/V^2. When we make these substitutions, equation (3) becomes $\alpha/V^2 = m/r^2$, whence

$$m = \alpha r^2/V^2. \tag{4}$$

With the second as the unit of time, the kilometer as the unit of distance, and using values of α and r for the surface of the sun, we find for m:

$$m = \frac{(0.2738 \text{ km.})(695600 \text{ km.})^2}{(299860 \text{ km.})^2} = 1.47 \text{ km.}$$

The value for m is kilometers, not square kilometers, or cubic kilometers. In this sense m is a one-dimensional quantity, a length, and can be expressed in any unit of length.[2]

4. The Relativity Deflection at the Limb of the Sun. In works on relativity,[3] the deflection of a ray of light is expressed in the form

$$D_0 = 4m/R, \tag{5}$$

where D_0 is the deflection of a ray just grazing the limb of the sun, R is the radius, and m the mass in relativity units.

The quantity m is evaluated in (4). For the surface of the sun this becomes, using the previous notation, $m = R^2G/V^2$. Substituting in (5) and expressing D_0 in seconds of arc, we have

$$D_0'' = 4RG/V^2 \tan 1''. \tag{6}$$

Comparing this with (2), the Newtonian deflection, we see that it is approximately double. In the denominator of (6) we have V, the velocity of light in a vacuum, free from gravitational influence. In the Newtonian expression (2), we have V_0, the velocity of light at the limb of the sun after travelling subject to the attraction of gravitation from a distant star. The relation between the two is $V_0 = V + \sqrt{2RG}$. Using for the astronomical constants values taken from the *American Ephemeris and Nautical Almanac*, we obtain:

$$D_0'' \text{ (Newtonian)} = 0''.8702,$$
$$(1/2)D_0'' \text{ (Relativity)} = 0''.8737.$$

[1] If α is the acceleration per second per second, and V is the velocity of light per second, the acceleration per second in $1/V$ seconds would be α/V; and the acceleration per $1/V$ seconds in $1/V$ seconds would be α/V^2.

[2] Using the day as the unit of time and the mean distance of the earth from the sun as the unit of distance:

$$m = \frac{2.959 \times 10^{-4} \times 1}{3003} = 9.85 \times 10^{-9} \times \textit{mean distance} = 1.47 \text{ km.} = 0.915 \text{ mile.}$$

[3] Eddington, *Mathematical Theory of Relativity*, p. 91.

These values are so nearly equal that one naturally wonders whether theoretically they should be exactly equal. Presumably not, for in this paper, the approximations in the derivation of the Newtonian deflection affect the seventh significant figure. Dr. F. D. Murnaghan, after deriving the relativity deflection using elliptic integrals, concludes that the form $D_0 = 4m/R$ is correct to five significant figures.[1] Since the results differ in the third significant figure, the difference can hardly be attributed to neglected quantities in the approximations.

5. The Deflection Near the Limb of the Sun. In practice, a star exactly at the limb of the sun cannot be seen. The relativity deflection for a distance r from the center of the sun, where the acceleration is g, is from (6)

$$D'' = 4rg/(V^2 \tan 1'').$$

From the law of inverse squares we have $g = R^2G/r^2$. Substituting and rearranging:

$$D'' = \frac{4RG}{V^2 \tan 1''} \cdot \frac{R}{r} = D_0'' \frac{R}{r}.$$

The expression for the Newtonian deflection, in terms of that at the limb, would obviously reduce in the same way to $D_0''(R/r)$.

This is a convenient form for use in the measurement of eclipse plates. Expressing r, the distance from the center of the sun, in units of the sun's semi-diameter, we have the form $D'' = D_0''/r$. The relativity deflection for each star would be $D'' = 1''.75/r$ and the Newtonian deflection, $D'' = 0''.87/r$.

NOTE ON NEUBERG'S CUBIC CURVE.[2]

By FRANK MORLEY, Johns Hopkins University.

Fixing points on a plane by means of the values of a complex variable, the squared distance λ_{12}' between two points a_1' and a_2' is $\lambda_{12}' = (a_1' - a_2')(\bar{a}_1' - \bar{a}_2')$, where $\bar{a}_1'$, $\bar{a}_2'$ denote, as usual, the conjugate complex numbers to a_1' and a_2' respectively. Starting out with three points a_1', a_2', a_3' in the plane, Neuberg's cubic is the locus of points a_4' satisfying the equation[3]

$$\Delta' \equiv \begin{vmatrix} \lambda_{23}'\lambda_{14}' & \lambda_{23}' + \lambda_{14}' & 1 \\ \lambda_{31}'\lambda_{24}' & \lambda_{31}' + \lambda_{24}' & 1 \\ \lambda_{12}'\lambda_{34}' & \lambda_{12}' + \lambda_{34}' & 1 \end{vmatrix} = 0.$$

[1] *Philosophical Magazine*, **43**, 580, 1922.

[2] When I called Professor Morley's attention to the unsolved problem proposed by Messrs. T. W. Moore and J. H. Neelley in this MONTHLY (*1925*, 246), he communicated to me the solution contained in this note. I have taken the liberty of amplifying somewhat his remarks so as to make the matter more easily intelligible to a wider circle of readers of the MONTHLY. F. D. MURNAGHAN.

[3] *Cf.* Brown, B. H. The 21-point Cubic. This MONTHLY (*1925*, 110–115).

Let us now subject the points a_1', a_2', a_3', a_4' to an inversion with center at the point a_5. If k is the radius of inversion, any point u' inverts into a point v found from the equation $(v - a_5)(\bar{u}' - \bar{a}_5) = k^2$. From this it follows at once that if we denote the points into which a_1', a_2', a_3', a_4' invert by a_1, a_2, a_3, a_4, respectively,

$$(a_1' - a_2')(\bar{a}_1' - \bar{a}_2') = k^4(a_1 - a_2)(\bar{a}_1 - \bar{a}_2)/(a_1 - a_5)(\bar{a}_1 - \bar{a}_5)(a_2 - a_5)(\bar{a}_2 - \bar{a}_5)$$

or

$$\lambda_{12}' = k^4\lambda_{12}/\lambda_{15}\lambda_{25}.$$

Proceeding similarly with the other squared distances we find

$$\Delta' = k^{12}I_2/\lambda_{15}^2\lambda_{25}^2\lambda_{35}^2\lambda_{45}^2, \quad \text{where} \quad I_2 \equiv \begin{vmatrix} \lambda_{23}\lambda_{14} & \lambda_{23}\lambda_{15}\lambda_{45} + \lambda_{14}\lambda_{25}\lambda_{35} & 1 \\ \lambda_{31}\lambda_{24} & \lambda_{31}\lambda_{25}\lambda_{45} + \lambda_{24}\lambda_{35}\lambda_{15} & 1 \\ \lambda_{12}\lambda_{34} & \lambda_{12}\lambda_{35}\lambda_{45} + \lambda_{34}\lambda_{15}\lambda_{25} & 1 \end{vmatrix}.$$

If we expand this determinant and use the notation (1 2 3 4 5) for the product $\lambda_{12}\lambda_{23}\lambda_{34}\lambda_{45}\lambda_{51}$, it is seen at once to be a sum of 12 terms of the type (1 2 3 4 5). Each of these terms is cyclic and reversible, *i.e.*,

$$(1\,2\,3\,4\,5) = (2\,3\,4\,5\,1) = (3\,4\,5\,1\,2) = (4\,5\,1\,2\,3) = (5\,1\,2\,3\,4)$$

and

$$(1\,2\,3\,4\,5) = (1\,5\,4\,3\,2),$$

etc. This reversibility reduces the $4! = 24$ cyclic arrangements of 5 things to 12 and it is these twelve that occur in the expansion. The sign to be attached to any term is + or − according as the arrangement is of the same class as (2 3 1 4 5) or not. For example, the sign attached to the term (1 5 2 3 4) is −. The important thing to notice is that I_2 involves the five points a_1, a_2, a_3, a_4, a_5 symmetrically. If we put $I_2 = 0$, we may regard this as the equation of the locus of a point a_4, the other four points a_1, a_2, a_3 and a_5 being given. The inverse of this locus in an inversion whose center at a_5 is the Neuberg curve of the three points a_1', a_2' and a_3'. We shall call, then, the curve whose equation is $I_2 = 0$ the inverted Neuberg curve with respect to one of the points (say a_5) of any three of the remaining four (say a_1, a_2 and a_3). The fifth point a_4 is the variable point which traces out the locus.

We shall now choose the point a_5 so that the points (a_1, a_2, a_3), into which the original triad (a_1', a_2', a_3') invert, are the vertices of an equilateral triangle. To see how this is done let us introduce, for the moment, homogeneous coördinates by writing the complex variable z in the form (z_1/z_2) and multiplying through by the proper power of z_2. To get back to the original complex variable, make $z_2 = 1$, $z_1 = z$. Then the triad (a_1', a_2', a_3') may be regarded as given by the zeros of the cubic polynomial

$$f'(z_1', z_2') \equiv \alpha_0'z_1'^3 + 3\alpha_1'z_1'^2z_2' + 3\alpha_2'z_1'z_2'^2 + \alpha_3'z_2'^3.$$

Any inversion is equivalent to a linear transformation of the form

$$z_1' = l\bar{z}_1 + m\bar{z}_2; \qquad z_2' = n\bar{z}_1 + p\bar{z}_2,$$

and we wish to find l, m, n and p so that (a_1, a_2, a_3) are the zeros of a polynomial $f(z_1, z_2)$ of the type $\alpha_0 z_1^3 + \alpha_3 z_2^3$, the middle terms being missing. Now the Hessian of f, *i.e.*, the quadratic polynomial

$$\begin{vmatrix} \frac{\partial^2 f}{\partial z_1^2} & \frac{\partial^2 f}{\partial z_1 \partial z_2} \\ \frac{\partial^2 f}{\partial z_1 \partial z_2} & \frac{\partial^2 f}{\partial z_2^2} \end{vmatrix},$$

has for its zeros in this instance $z_1 = 0$ and $z_2 = 0$, *i.e.*, the points 0 and ∞. Since the Hessian of $\bar{f}' = (\bar{l}\bar{p} - \bar{m}\bar{n})^{-2}$ times the Hessian of f, it follows that the triad a_1', a_2', a_3' and its Hessian points (h_1', h_2') invert into a triad (a_1, a_2, a_3) and its Hessian points (h_1, h_2). Conversely if we can fix the Hessian points h_1 and h_2 of a triad at 0 and ∞, we know the triad is equilateral (for the Hessian points of $\alpha_0 z_1^3 + 3\alpha_1 z_1^2 z_2 + 3\alpha_2 z_1 z_2^2 + \alpha_3 z_2^3$ are given by

$$(\alpha_0\alpha_2 - \alpha_1^2)z_1^2 + (\alpha_0\alpha_3 - \alpha_1\alpha_2)z_1 z_2 + (\alpha_1\alpha_3 - \alpha_2^2)z_2^2 = 0$$

and we are therefore given that $\alpha_0\alpha_2 - \alpha_1^2 = 0$ and $\alpha_1\alpha_3 - \alpha_2^2 = 0$. If $\alpha_0 \neq 0$, this forces $\alpha_1 = 0$, $\alpha_2 = 0$ so that the triad is equilateral). If then we invert the original triad (a_1', a_2', a_3') from one of its Hessian points h_1', the triad will invert into an equilateral triad and the other Hessian point h_2' will invert into its center $h_2 = 0$.[1] Since a circle and a pair of inverse points invert into a circle and a pair of inverse points, we see that the Hessian points of any triad are inverse points with respect to the circumcircle of the triad. Thus the circumcenter and the two Hessian points lie on a line.

Since we are now separating out the variable point a_4 on the locus and the center of inversion a_5 from the five points which enter symmetrically the expression I_2, we write out I_2 in the form

$$\begin{aligned} I_2 = {} & (\lambda_{23}\lambda_{31}\lambda_{45} + \lambda_{34}\lambda_{35}\lambda_{12})(\lambda_{14}\lambda_{52} - \lambda_{15}\lambda_{42}) \\ & + (\lambda_{31}\lambda_{12}\lambda_{45} + \lambda_{14}\lambda_{15}\lambda_{23})(\lambda_{24}\lambda_{53} - \lambda_{25}\lambda_{43}) \\ & + (\lambda_{12}\lambda_{23}\lambda_{45} + \lambda_{24}\lambda_{25}\lambda_{31})(\lambda_{34}\lambda_{51} - \lambda_{35}\lambda_{41}), \end{aligned}$$

where each term follows from the preceding one by a cyclic interchange of 1, 2 and 3. We further denote the variable point a_4 on our locus by x and the fixed center of inversion by y.

The points (a_1, a_2, a_3) now form an equilateral triad whose center is at the origin and we may so choose the radius of the inversion whose center is at $a_5 = y$

[1] A figure readily shows that this is equivalent to saying that the Hessian points of a triad are the points whose pedal triangles, with respect to the triangle formed by the triad, are equilateral.

that the three points (a_1, a_2, a_3) lie on the unit circle whose center is at the origin.[1] We have then $a_1 = t$, $a_2 = \omega t$, $a_3 = \omega^2 t$, where ω, ω^2 are the complex cube roots of unity and t is a turn, *i.e.*, a complex number whose conjugate is its reciprocal. It follows at once that

$$\lambda_{12} = \lambda_{23} = \lambda_{31} = 3, \qquad \lambda_{45} = (x - y)(\bar{x} - \bar{y}),$$
$$\lambda_{14} = 1 + x\bar{x} - t\bar{x} - (x/t), \qquad \lambda_{15} = 1 + y\bar{y} - t\bar{y} - (y/t),$$

and the expressions for λ_{24}, λ_{34}, etc., follow on replacing t by ωt and $\omega^2 t$, respectively, in these. The first of the three terms in the expression for I_2 is accordingly $3(3\lambda_{45} + \lambda_{34}\lambda_{35})(\lambda_{14}\lambda_{52} - \lambda_{15}\lambda_{42})$ and we have at once

$$\lambda_{34}\lambda_{35} = (1 + x\bar{x})(1 + y\bar{y}) + x\bar{y} + y\bar{x} + (\omega^2 xy/t^2) + \omega t^2 \bar{x}\bar{y}$$
$$- \{x(1 + y\bar{y}) + y(1 + x\bar{x})\}(\omega/t) - \{\bar{x}(1 + y\bar{y}) + \bar{y}(1 + x\bar{x})\omega^2 t,$$
$$\lambda_{14}\lambda_{52} - \lambda_{15}\lambda_{42} = (\omega^2 - \omega)[\{\bar{x}(1 + y\bar{y}) - \bar{y}(1 + x\bar{x})\}\omega^2 t$$
$$+ \{y(1 + x\bar{x}) - x(1 + y\bar{y})\}(\omega/t) + x\bar{y} - y\bar{x}].$$

In developing the product $(3\lambda_{45} + \lambda_{34}\lambda_{35})(\lambda_{14}\lambda_{52} - \lambda_{15}\lambda_{42})$ we need not consider the terms in t, t^{-1}, t^2, t^{-2} since in each of these t has to be replaced by ωt, $\omega^2 t$ in turn, the resulting expressions then being added to the original one, and since $1 + \omega + \omega^2 = 0$. We find, then,

$$I_2 = 9(\omega^2 - \omega)\{A + (B/t^3) - \overline{B}t^3\},$$

where

$$A = \{3(x - y)(\bar{x} - \bar{y}) - (1 + x\bar{x})(1 + y\bar{y}) + x\bar{y} + \bar{x}y\}(x\bar{y} - y\bar{x}),$$
$$B = xy^2(1 + x\bar{x}) - yx^2(1 + y\bar{y}).$$

The Apollonian circles of any triad are the three circles one through each point *about* the other two, *i.e.*, having these two as inverse points. Hence a triad and its Apollonian circles inverts into a triad and its Apollonian circles. In the case of the equilateral triad the Apollonian circles are the straight lines one through each point bisecting perpendicularly the join of the other two. The three remaining points where the Apollonian circles cross the circumcircle of the triad form the "counter-triad" and in the case of the equilateral triad given by the polynomial $x^3 - t^3 = 0$ the counter-triad is given by $x^3 + t^3 = 0$. This counter-triad has the same circumcircle and Hessian points as the original triad, a result holding, therefore, for every triad and its counter-triad. The equation of the inverted Neuberg cubic for the original triad being

$$I_2 = 0 \quad \text{or} \quad A + (B/t^3) - \overline{B}t^3 = 0,$$

that of the inverted cubic for the counter-triad is $A - (B/t^3) + \overline{B}t^3 = 0$ and

[1] It is easy to see that if d_1 and d_2 are the distances of the circumcenter and the second Hessian point from the center of inversion and R is the radius of the circumcircle, then $k^2 = d_1 d_2 / R$.

upon adding these we see that the curve $A = 0$ goes through their points of intersection. Now $A = 0$ when either of its factors $x\bar{y} - y\bar{x}$ or $3(x - y)(\bar{x} - \bar{y}) - (1 + x\bar{x})(1 + y\bar{y}) + x\bar{y} + y\bar{x}$ is zero. The first factor gives the straight line joining the origin to the center of inversion, *i.e.*, the straight line through the center of inversion and the Hessian points of the equilateral triad. The other factor gives the circle whose equation is

$$C \equiv 2(x - y)(\bar{x} - \bar{y}) - 1 - y\bar{y}x\bar{x} = 0.$$

It will now be convenient to take the axis of reals through the center of inversion so that y is a real number μ. If we look at the expression for B, we see that $B - \overline{B} = \mu(x - \bar{x})\{\mu(1 + x\bar{x}) - (x + \bar{x})(1 + \mu^2)\}$ and $A = \mu(x - \bar{x})C$. Hence in the special cases $t^3 = \pm 1$ the inverted Neuberg curve degenerates, one of the factors giving the axis of reals and the other factor giving the circles $C \pm C_1 = 0$, where $C_1 \equiv \mu(1 + x\bar{x}) - (x + \bar{x})(1 + \mu^2)$. If we suppose these degenerate inverted Neuberg curves, $C \pm C_1 = 0$, known, then the circle $C = 0$ is described as belonging to the pencil determined by them and harmonic to the circle $C_1 = 0$. It is seen at once that $C_1 = 0$ goes through the points $-\omega\mu$ and $-\omega^2\mu$ and it is orthogonal to the circumcircle $x\bar{x} - 1 = 0$. It is readily seen that the two points $-\omega\mu$, $-\omega^2\mu$ invert into the vertices of the two equilateral triangles described upon the join of the two Hessian points of the original triad. Hence the inverse of the circle $C = 0$ is the circle through the two vertices of these equilateral triangles and orthogonal to the circumcircle.

The geometrical significance of the relations $t^3 = \pm 1$ follows at once from a consideration of the expressions given for λ_{12}', etc. If $t^3 = \pm 1$, two of the three squared distances λ_{15}, λ_{25}, λ_{35} are equal and so two of the original squared distances λ_{23}', λ_{31}', λ_{12}' are equal, *i.e.*, the original triangle is isosceles. In this case the Neuberg cubic reduces to a circle and a line, namely, the join of the Hessian points (which is the perpendicular bisector of the base of the isosceles triangle). As we vary the t in the equation $x^3 - t^3 = 0$ of our equilateral triad we get a series of inverse triads having a common circumcircle and Hessian points. Out of all these triads two are isosceles and these furnish, through their degenerate Neuberg cubics, the two circles which are the inverses of the pair $C \pm C_1 = 0$ and which determine the pencil to which the sought for circle belongs.[1]

[1] It may be mentioned that the Neuberg cubic of a triangle may be most easily studied starting from the property that it passes through the isogonal conjugate of each of its points and that the join of each such pair of isogonal conjugates is parallel to the Euler line of the triangle. Mr. B. C. Patterson, one of Dr. Morley's students, has studied the covariants of the Neuberg cubic under the inversion group.

APPLICATION OF THE THEOREM OF RESIDUATION TO THE 21-POINT CUBIC.

By O. M. THALBERG, Oslo, Norway.

With great interest I have read the paper by B. H. Brown in the MONTHLY, *1925*, pp. 110–115. Perhaps it may prove instructive to see the application of the Brill-Noether theorem [1] of residuation to the 21-point cubic or to any cubic of the pencil (through A_i, K, K_i and the two circular points) considered by Mr. Brown (the notation refers to the figure, *l.c.*, p. 111, which is here reproduced).

Since the points A_i, K, K_i are collinear and the two circular points and the points A_j, A_k, K, K_i are concyclic, the point A_i must be coresidual with the pointgroup consisting of the two circular points and A_j, A_k, and thus we may write down the following

THEOREM: *Any straight line through A_i meets the cubic again in two points concyclic with A_j and A_k.*

As special cases of this theorem we at once deduce the following ones:

If the circumcircle of the triangle $A_1A_2A_3$ again meets the cubic in I, then IA_i is tangent to the cubic at A_i (*cf.* Brown, p. 112).

A circle exists touching the cubic at K and at K_i. There exists also a circle passing through J_k and A_k and touching the cubic at A_j.

The line A_iJ_i is parallel to the real asymptote.

The point K is coresidual to the pointgroup consisting of the two circular points and A_i, K_i (the two points A_j, K_j forming the common residual points). Hence:

THEOREM: *Any straight line through K meets the cubic again in two points concyclic with A_i and K_i.*

In the same way we obtain the following

THEOREM: *Any straight line through K_i meets the cubic again in two points concyclic with A_j and K_k as well as with A_i and K.*

As special cases of these theorems the following ones are of particular interest:

The tangents at K_i (K) are parallel to the real asymptote (*cf.* Brown, p. 112).

There exists a circle touching the cubic at A_i and K_i, or at A_j and K_k, or at A_i and K.

As the third (real) cubic point at infinity is coresidual to the pointgroup K, K_i, A_j, A_k as well as to the pointgroup A_i, A_j, K_i, K_j (the two circular points being the common residual points), we obtain the following

THEOREM: *Any line parallel to the real asymptote meets the cubic again in two points, lying on a conic through the pointgroup K, K_i, A_j, A_k as well as on a conic through the pointgroup A_i, A_j, K_i, K_j.*

From this follow several special theorems, of which we mention only the following:

[1] See *Math. Annalen*, vol. 7, 1874, pp. 272–3.

A_i, J_i, A_j, A_k, K_j and K_k lie on the same conic.

A_i, J_i, K, K_i, A_j and A_k lie on the same conic.

There exists a conic through K, K_i, A_j, A_k touching the cubic at K_j, and a conic through A_i, A_j, K_i, K_j touching the cubic at K_k.

The two circular points at infinity are coresidual to A_i, J_i as well as to twice the point $K_i(K)$ (the third cubic point at infinity is the common residual point). Hence:

THEOREM: *Any circle cuts the cubic again in four points lying on a conic through A_iJ_i as well as on a conic touching the cubic at $K_i(K)$.*

By the theorem of residuation we further derive the following

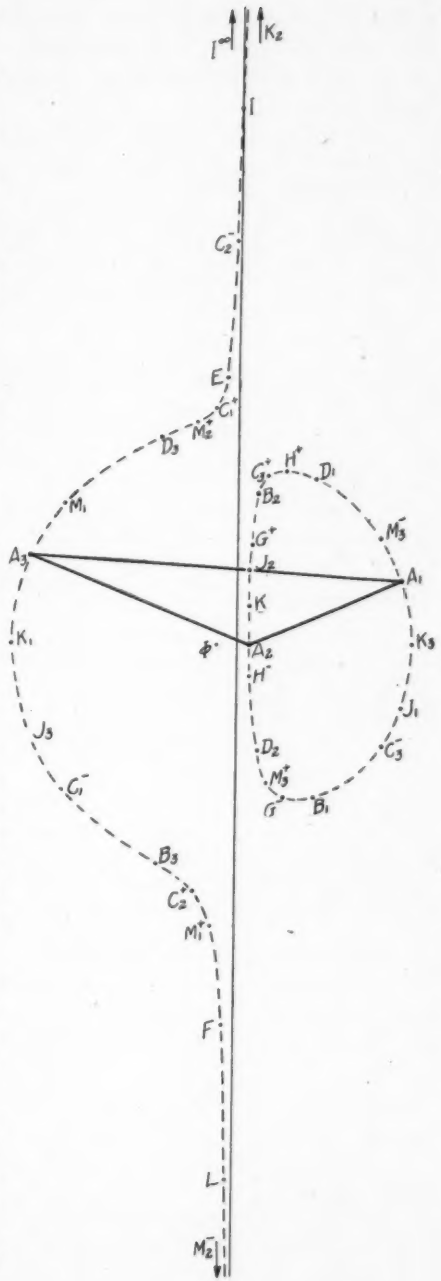
THEOREM: *Any circle through A_i, K_j cuts the cubic again in two points lying on a circle through A_j, K_i as well as on a circle through A_k, K.*

Finally we wish to show that it is convenient here to make use of the following well-known theorem:

Let T_1, T_2, T_3 and T_4 be the four points of contact of the four tangents from the point P on the cubic. Then the three diagonal points S_1, S_2 and S_3 of the quadrangle $T_1T_2T_3T_4$ lie on the cubic, and these three points and the point P have the same tangential point.

From this theorem and from the above property of the line A_iJ_i [or from the property of the tangent at $K_i(K)$] now the following theorem follows:

The common tangential I of the points A_i is the third intersection of the real asymptote with the cubic. The three points J_i and the point I have the same tangential point L (cf. Brown, pp. 112, 114). *The three diagonal*



points of the quadrangle $J_1J_2J_3I$ lie on the cubic and have common tangential point with L.

All the above theorems relate to any cubic of the pencil considered by Mr. Brown.

If we particularly consider the 21-point cubic, the theorem of residuation in connection with the results in Mr. Brown's paper will give us several new theorems as for instance the following:

Any circle cuts the cubic again in four points, lying on a conic through B_i, D_i —on a conic through C_i^*, M_i^*—on a conic through E, F as well as on a conic through G^*, H^* (*cf.* Brown, p. 115).

AMERICAN CONTRIBUTIONS TO MATHEMATICAL SYMBOLISM.

By FLORIAN CAJORI, University of California.

1. Maya Number-System. Probably seven or eight centuries prior to the introduction of the zero in the Hindu-Arabic numerals, the Maya of Central America had a fully developed number system on the scale of 20 (except in one step). This system, as it appears in Maya codices, had a symbol for zero, and an extended application in the wonderful system of Maya chronology.[1]

2. Peruvian Knot Records. The use of knots in cords for reckoning, and recording numbers, early practised by the Chinese, had a most remarkable development among the Inca of Peru, from the eleventh to the sixteenth century of our era. Upon a twisted woolen cord (quipu) other smaller cords of different colors were tied. The color, length and number of knots, and distance of one from the other, all had their significance.[2] Quipu-like string records have been found in North America among the Indians of the Northwest.[3]

3. Dollar Mark. An extended study of manuscripts has led to the conclusion that our dollar mark descended, during the last quarter of the eighteenth century, from the Spanish-American abbreviation "p^s" for "pesos."[4]

4. Sporadic Notations for Radicals. In an anonymous publication, *The Columbian-Arithmetician. By an American*, Haverhill, Mass., 1811, there is added to the usual exponental notation 4^2, 2^m the following bold innovation (p. 13): $^2 4$ to mean $\sqrt{4}$, $^3 8$ to mean $\sqrt[3]{8}$, $^m 8$ to mean $\sqrt[m]{8}$. This symbolism found no favor.

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[1] See S. G. Morley, *An Introduction to the Study of the Maya Hieroglyphs.* Government Printing Office, Washington, 1915.

[2] L. Leland Locke, *The Ancient Quipu or Peruvian Knot Record.* 1923.

[3] J. D. Leechman and M. R. Harrington, *String Records of the Northwest, Indian Notes and Monographs,* 1921.

[4] F. Cajori in *Popular Science Monthly,* vol. 81, 1912, p. 521; *Science,* N.S., vol. 38, 1913, p. 848.

5. B. Peirce's Signs for our π, e and i. These are shown [1] in Figs. 1 and 2; they were used by his sons, J. M. Peirce and C. S. Peirce.

NOTE ON TWO NEW SYMBOLS.

BY BENJAMIN PEIRCE,
Professor of Mathematics in Harvard College, Cambridge, Mass.

THE symbols which are now used to denote the Neperian base and the ratio of the circumference of a circle to its diameter are, for many reasons, inconvenient; and the close relation between these two quantities ought to be indicated in their notation. I would propose the following characters, which I have used with success in my lectures:—

π to denote ratio of circumference to diameter,
e to denote Neperian base.

It will be seen that the former symbol is a modification of the letter c (*circumference*), and the latter of b (*base*).

The connection of these quantities is shown by the equation,

$$\pi^e = (-1)^{-\sqrt{-1}}.$$

FIG. 1. B. Peirce's signs for π and e in the *Mathematical Monthly*, 1859.

$$\sqrt{e^{\pi}} = \sqrt[i]{i}$$

FIG. 2. From J. M. Peirce's *Tables*, 1871.

6. Equivalence in Geometry. This was expressed by the sign $\backsimeq$ which occurs in C. Davies' *Elements of Geometry and Trigonometry*,[2] 1851. For a quarter of a century following 1885, the sign enjoyed considerable popularity; the two signs $=$ and $\backsimeq$ were used in Geometry to express congruence and equivalence.

7. Approaching the Limit was designated by $\doteq$. The sign is due to J. E. Oliver of Cornell who at first used it in the sense "is nearly equal to." In 1880, W. E. Byerly [3] gave it the meaning "approaches as a limit." This same symbol $\doteq$ was used in 1875 by A. Steinhauser [4] of Vienna in the sense "nahezu gleich," but to the best of our knowledge Oliver is in no way indebted to Steinhauser.

8. A Symbolism in Vector Analysis was invented by Josiah Willard Gibbs of Yale, a pioneer in this field. In 1881 he marked a scalar or "direct" product by $\alpha \cdot \beta$, a vector or "skew" product by $\alpha \times \beta$, where small Greek letters represent

[1] B. Peirce in Runkle's *Mathematical Monthly*, vol. 1, No. 5, 1859, pp. 167, 168, "Note on two new symbols."

EDITOR'S NOTE:—Professor W. E. Story of Clark University used these symbols in this sense in 1907 in a course of lectures on the calculus of finite differences. NORMAN ANNING.

[2] Charles Davies, *Elements of Geometry and Trigonometry*, from the works of A. M. Legendre, New York, 1851, p. 87.

[3] W. E. Byerly, *Elements of the Differential Calculus*, Boston, 1880, p. 7. Oliver himself used the symbol in print in the *Annals of Mathematics*, Charlottesville, Va., vol. 4, 1888, pp. 187, 188. It is used also in Oliver, Wait and Jones' *Algebra*, Ithaca, 1887, pp. 129, 161.

[4] A. Steinhauser, *Lehrbuch der Mathematik*. Algebra. Vienna, 1875, p. 292.

vectors. He let a_0 stand for the magnitude of vector α. He[1] marked triple products, $\alpha \times \beta \cdot \gamma$, $(\alpha \cdot \beta)\gamma$, $\alpha[\beta \times \gamma]$.

9. Symbolic Logic was developed by C. S. Peirce. "He understood how to profit by the work of his predecessors, Boole and De Morgan, and built upon their foundations, and he anticipated the most important procedures of his successors even when he did not work them out himself."[2] C. S. Peirce[3] introduced a considerable number of new symbols, for instance, $-<$ for "inclusion in" or "being as small as"; x, y signifies commutative multiplication; in multiplication, "identical with—" is *1*; etc. Other symbols were proposed for symbolic logic by Mrs. Christine Ladd Franklin and O. H. Mitchell.[4]

10. General Analysis. Of notations introduced in America in the present century, I mention only the symbolism used by E. H. Moore in his general analysis.[5] He takes some of his logical signs from Peano's *Formulario mathematico*, 1906, and uses them approximately in the sense of Peano. Among Moore's other signs are $\neq$ for logical diversity, $\equiv$ for definitional identity, $\ni$ for "such that," x^P for "x has the property P." Moore's aim is different from that of Peano, Whitehead and Russell whose object was to proceed with absolute certainty in difficult, abstract studies of the foundations of mathematics, and who for that purpose used elaborate notations. Moore aims to meet the needs of the working mathematicians who consider extreme logical complications relatively unimportant, but desire simplicity and flexibility of notation.

None of the ten notations cited as originating in America has thus far found general acceptance in Europe, except, perhaps, the dollar mark.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. DERIVATION OF ANNUITY FORMULAS WITHOUT SERIES.

BY F. R. MORRIS, Fresno State College.

It is customary in deriving the annuity formulas to treat an annuity as a geometrical progression and to use the formula for the sum of n terms of this

[1] *The Scientific Papers of J. Willard Gibbs*, vol. 2, 1906, pp. 22, 23.

[2] C. I. Lewis, *A Survey of Symbolic Logic*, Berkeley, 1918, p. 79.

[3] C. S. Peirce, "On an Improvement in Boole's Calculus of Logic" in *Proceedings of the American Academy of Arts and Sciences*, vol. 7, Cambridge and Boston, 1868, pp. 250–261; "Description of a Notation for the Logic of Relatives" in *Memoirs Amer. Acad. of Arts and Sciences*, Cambridge and Boston, N.S., vol. 9, 1867, pp. 317–378.

[4] See *Studies in Logic*. By Members of the Johns Hopkins University. Boston, 1883.

[5] E. H. Moore, Introduction to a Form of General Analysis. *New Haven Mathematical Colloquium*. New Haven, 1910, p. 150.

series.[1] However, it is not necessary in the development of these formulas to make use of the geometrical progression, and it is the purpose of this paper to give a development based upon the definitions of annuities and compound interest.

The compound interest law states that $(1 + i)^n$ is the amount on 1 for n years at rate i. It is easy to see that the application of this law gives results which are in agreement with those commonly used, providing n is zero or a positive integer. If n is a negative integer, the function gives the value of 1 n years before it is due. Then *by definition* $(1 + i)^n$ is said to be the value of 1 for any real value of n.[2]

If the interest is converted into principal more than once each year, n may be considered as the number of periods and i the rate per period; but, to avoid confusion in this case, it is better to introduce new symbols and definitions. Let m be the number of conversion periods per year. The ratio of the entire interest for a year to the investment at the beginning of that year is called the effective rate of interest and is designated by i. The nominal rate is defined as m times the ratio of the interest for a period to the investment at the beginning of that period, which is usually the rate named in the document, and is designated by j. Hence j/m is the rate per period and $(1 + j/m)^m$ is the amount on 1 for one year. But $1 + i$ is also the amount on 1 for one year. Therefore

$$1 + i = (1 + j/m)^m. \tag{1}$$

An ordinary annuity may be defined as a number of equal payments made at the end of equal periods. It is convenient to use as a basis the annuity for which the payment is 1 at the end of each year. The annual payment is called the annual rent. If the rent is not 1, results may be easily obtained by multiplying the corresponding results for the basic annuity by the given rent. The amount of an annuity of 1 per year at the end of n years where the payments bear interest at the uniform effective rate i is designated by the symbol $s_{\overline{n}|}$. It is desirable to express this amount as a function of i and n.

The income of 1 at rate i is an ordinary annuity the annual rent of which is i. The amount of this annuity at the end of n years is the compound interest on 1 for n years, which is $(1 + i)^n - 1$. If this expression is divided by i, the quotient is the amount of an annuity the annual rent of which is 1 instead of i. As a result we have the fundamental formula

$$s_{\overline{n}|} = \{(1 + i)^n - 1\}/i. \tag{2}$$

The next problem is to derive the formula for the amount of an annuity of 1 per year paid in p equal installments. This amount is given the symbol $s^{(p)}_{\overline{n}|}$.

[1] For an exception, see the Discussion by C. N. Reynolds, Jr., in this MONTHLY (*1922*, 122). —EDITOR.

[2] Although this definition is generally used in the theory of investment, it does not quite agree with the common practice of computing compound interest for the integral number of years and simple interest for the remaining fraction of a year.

It is evident that the p payments, each of which is $1/p$, will have a value greater than 1 at the end of the year. We may think of 1 as the nominal annual rent, while the value of the p payments at the end of the year is the effective annual rent. Then $s^{(p)}_{\overline{n}|}$ is the product of $s_{\overline{n}|}$ and the effective annual rent.

The effective annual rent may be obtained from equation (1). Let m be replaced by p. The rent per period which will produce i in one year is j/p. Hence $1/p$ per period will produce an annual income of i/j, which is the effective annual rent. From the same equation j is found to be $p[(1+i)^{1/m}-1]$. Therefore the desired formula is

$$s^{(p)}_{\overline{n}|} = \frac{i}{p((1+i)^{1/m}-1)} \cdot s_{\overline{n}|}. \tag{3}$$

We see from the compound interest law that multiplying any amount by $1+i$ gives the accumulated value a year later, and that dividing by $1+i$ gives the value a year in advance. Making use of this principle we obtain the present value of the basic annuity. The present value, which is denoted by $a_{\overline{n}|}$, is the value of an annuity which n years later has a value $s_{\overline{n}|}$. Hence

$$a_{\overline{n}|} = (1+i)^{-n} s_{\overline{n}|} = \frac{1-(1+i)^{-n}}{i}. \tag{4}$$

Likewise we obtain the formula for $a^{(p)}_{\overline{n}|}$ which is the present value with p installments per year:

$$a^{(p)}_{\overline{n}|} = \frac{i}{p((1+i)^{1/m}-1)} \cdot a_{\overline{n}|}. \tag{5}$$

By this same principle the various formulas for annuities due and for deferred annuities may be derived.

If the interest is converted more than once a year, the annuity formulas are modified by substituting for i the value given by equation (1), thus giving functions of j, m, n and p. For the case $m = p$, formulas (3) and (5) reduce to forms similar to formulas (2) and (4). This completes the system of essential formulas for annuities.

II. Errata in Steinhauser's 20-Place Logarithm Table.

By E. B. Escott, Oak Park, Illinois.

One of the most convenient tables for computing logarithms to 20 places is Anton Steinhauser's *Hilfstafeln zur präcisen Berechnung zwanzigstelliger Logarithmen zu gegebenen Zahlen und der Zahlen zu 20stelligen Logarithmen* (Wien, 1880). In the first edition, a list of 46 errata was given and about five years later an additional list of 17 errata was given. Although the author says that he is convinced that the table contains no more errata, the following additional errata will show that this statement is far from being true. It is to be hoped

TABLE I.

COLUMNS *A*.

Page	Log. of	Figure Group	Instead of	Should be	Page	Log. of	Figure Group	Instead of	Should be
3	1153	3	9294	7294	59	6718	3	9405	9505
3	1192	4	7624	7625	60	6884	3	2655	1655
5	1302	3	6232	4232	62	7009	4	5606	5605
8	1635	4	1755	1775	62	7009	3	0083	9983
8	1697	1	9242	9742	63	7173	3	0825	0835
17	2536	4	3924	4924	64	7249	4	7849	7809
18	2658	4	7497	5497	66	7467	3	1428	1328
23	3108	3	0828	0128	66	7496	3	7779	7879
24	3247	3	8726	8626	69	7772	3	1727	1927
26	3442	3	6991	5991	69	7778	3	8711	8811
29	3748	3	2115	2215	70	7810	5	8426	8926
30	3886	3	6163	6263	71	7916	3	5434	5534
33	4187	3	0691	0891	72	8046	3	8135	8235
35	4304	3	2558	2658	73	8136	3	9814	9914
36	4402	3	8117	8217	75	8374	3	6355	6555
37	4534	3	5890	5790	75	col. *n*		5310	8310
40	4884	4	5565	7565	78	8608	3	8222	8322
44	5256	3	6451	6551	78	8662	3	9493	9393
44	5292	3	5715	5515	79	8786	3	8829	8929
45	5304	3	5496	5396	80	8804	3	3781	3881
46	5427	3	1679	1579	81	8972	3	4810	4910
47	5516	3	7703	7503	81	8974	3	5297	5197
47	5526	3	0680	0580	82	9068	3	1554	1454
54	col. *n*		3255	6255	83	9166	3	2808	2908
	"		3265	6265	83	9175	3	0224	2924
	"		3275	6275	84	9202	3	8828	8928
	"		3285	6285	85	9363	3	3484	3384
	"		3295	6295	86	9416	3	9735	9835
54	6216	3	6492	5792	86	9456	3	3437	3537
54	6269	4	8926	9826	86	9496	3	5846	5946
55	6319	3	5463	5363	87	9566	3	6435	6535
55	6394	3	1835	1935	87	9574	3	3045	3145
56	6494	3	4389	4289	89	9741	3	3445	3345

TABLE II.

COLUMNS *B*.

Page	Log. of	Figure Group	Instead of	Should be	Page	Log. of	Figure Group	Instead of	Should be
100	0321	1	9389	9398	240	7305	3	6399	6299
148	2716	4	94	93	240	7306	3	9796	9696
148	2717	4	99	98	240	7307	3	3194	3094
148	2718	4	1803	1802	240	7308	3	6592	6492
148	2719	4	07	06	240	7309	3	9990	9890
160	3313	3	7033	7933	240	7327	3	0048	1048
160	3313	last	7−	7	241	7392	4	90	91
166	3624	4	36	35	249	7795	4	3839	3840
191	4888	3	1377	1277	264	8505	5	4	3
193	4980	5	1	2	272	8911	4	72	82
197	5197	4	60	64	275	9065	5	4	3
222	6411	4	53	33	276	9103	3	0437	8437
229	col. *n*		7690	6790	288	9730	5	3	4
233	6983	3	1900	2000					

that the table may be checked against the manuscript table to 28 places of Dr. Edward Sang, which gives logarithms as far as the number 20,000, and which is in the possession of the Royal Society of Edinburgh.

It will be noticed that the errata in the logarithms in Column *A* are largely in the 10th figure, and most of these were discovered by comparing them with the 10 place logarithms in Vega's *Thesaurus*. Many of the errata in the logarithms in Columns *B*, *C* and *D* were discovered by comparing logarithms in the three columns on the same line. The differences in these logarithms would be about the same and any variation was investigated. For this reason, it is hoped that most of the errata in the first 10 figures of the logarithms in Columns *A* and the first 7 or 8 figures in the logarithms in Columns *B*, *C* and *D* have been found.

TABLE II.

COLUMNS *C*.

Page	Log. of	Figure Group	Instead of	Should be	Page	Log. of	Figure Group	Instead of	Should be
101	0381	3	6629	6619	160	3346	3	34	31
101	0382	3	0059	0049	179	4291	3	3758	5758
101	0383	3	3488	3478	214	6033	3	9353	9853
101	0384	3	6918	6908	229	6755	3	9337	9336
101	0385	3	0347	0337	229	6756	3	41	40
101	0386	3	3776	3766	229	6757	3	46	45
101	0387	3	7206	7196	229	6758	3	50	49
101	0388	3	0635	0625	229	6759	3	54	53
101	0389	3	4065	4055	232	6929	3	97	92
135	2053	3	11	16	254	8025	4	2	3

TABLE II.

COLUMNS *D*.

Page	Log. of	Figure Group	Instead of	Should be	Page	Log. of	Figure Group	Instead of	Should be
99	0266	2	85	55	208	5729	3	7298	7308
115	1066	3	5891	5791	219	6289	2	62	12
121	1399	2	25	75	231	6859	3	5285	2585
128	1707	3	4067	4068	242	7445	3	2	3
130	1817	3	1207	1307	256	8145	3	2	3
139	2296	3	4003	4013	262	8420	3	8	3
179	4291	1	2762	5762	270	8809	2	56	57
186	4600	3	7461	5461	276	9115	2	3585	9585
191	4895	3	1	2	282	9405	3	0390	3960
203	5455	3	7239	7639	286	9611	3	0406	0426
204	5546	2	95	85	291	9887	3	6054	6954

xii	$N = 71$	1	$3202 -$	$2302 -$
263	col. 1	line 1	$(11)(7)$	$(7)(11)$

The figure groups in the above are numbered from the right.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Cours complet de Mathématiques Spécielles, Vol. II, Geometrie. By J. HAAG. Paris, Gautier-Villars, 1921. 651 pages.

Volume I of this work, entitled *Algèbre et analyse,* appeared in 1914 and the author explains that the delay in the appearance of the second volume was due to the war. This book is divided into two parts, the first treating general problems in geometry, while the second part is concerned wholly with conics and quadrics. The method used is principally analytic but the author does not hesitate to use synthetic methods when it seems to him desirable to do so. Plane and solid geometry are treated in close proximity to each other. We find one chapter entitled the straight line in the plane while the next is entitled the plane in space, and so for many other topics.

The presentation is clear, concise and appealing. The arrangement is admirable. It has long been considered in this country that conic sections should be the first subject in geometry to present to the student. This course usually precedes calculus. Here we find Professor Haag presenting a course in geometry after calculus and placing the study of conics and quadrics at the end. Most of the elementary theorems of geometry are presented. I shall not attempt to give a detailed discussion of the contents; but in order to give an idea of its scope, shall mention a few chapters.

In Part I, chapter VIII, on vectors, treats sums, differences, scalar and vector products of vectors. Chapter IX is on anharmonic ratio and involution; chapter X, on trilinear and tetrahedral coördinates. Chapter XIV presents a study of a curve in the neighborhood of a point. Both plane and twisted curves are considered and the appearance of the latter when looked at along the tangent, normal or binormal, is given. Chapter XVIII is on unicursal curves and surfaces; chapter XXIII, on curvature of lines traced on a surface; chapter XXIV, on geometric problems which lead to differential equations; chapter XXVI, on ruled and developable surfaces; chapter XXVII, on systems of lines; chapter XXVIII, on transformations. Part II comprises 220 pages, and is all devoted to curves and surfaces of the second order.

C. L. E. MOORE.

Die Geburt der modernen Mathematik. II. Die Infinitesimalrechnung. By HEINRICH WIELEITNER. Karlsruhe in Baden, G. Braun, 1925. 72 pages. Price, 1 gold mark.

The second part of Dr. Wieleitner's little essay *Die Geburt der modernen Mathematik* carries out the policy of the first brochure. It consists of six chapters: the first relating to the introduction of the infinite series into elementary mathe-

matics; the second, to the concept of limits and of integrals; the third, to the development of the integral concept; the fourth, to the development of the differential calculus; the fifth, to the Leibniz and Newton contributions and to the dispute concerning priority; and the sixth, to a general survey of the problem.

There is a brief bibliography of works on the history of the calculus, and a valuable index of names, with the dates. With respect to the latter the reader will have to use caution in following the dates given. Some are questioned, which is quite proper, but others, which are equally doubtful, are stated as if they were certain. The reader will also need to observe that Dr. Wieleitner has not followed any international scheme of spelling of proper names, and, indeed, seems not to have read the proof with his usual care, as witness the name of Diophantus.

As to the work itself, it accomplishes in a satisfactory manner the purpose which the author and editors have in mind; namely, the setting forth of the critical points in the development of the infinitesimal calculus. The reader will find here in brief form a statement of the steps that were taken in reaching the stage which may be characterized by the name of Cavalieri, and then the subsequent stages from there on to the completion of the work in the period of Leibniz. He will also find a subsequent statement of the priority question with respect to Leibniz and Newton, a question which probably would never have arisen if it had not been for certain busybodies who stirred up the strife. As a brief summary the pamphlet will be of considerable service.

DAVID EUGENE SMITH.

Studies in Human Biology. By R. PEARL. Baltimore, Williams and Wilkins, 1924. 650 pages, 123 charts. Price $8.00.

This is essentially a volume of collected papers, as only a very few of the chapters have not previously appeared in the periodical literature. A review of this book in a sense is a review of the human activities of one biologist. Raymond Pearl is Professor of Biometry and Vital Statistics in the School of Hygiene and Public Health of the Johns Hopkins University. Consequently we find, as we would expect, that this book is limited to those phases of human biology that lend themselves well to statistical analysis and have bearing on problems of health and the questions that may be answered from the data of vital statistics. A glance through the table of contents shows that such a delimitation still permits of the study of almost all of the aspects of human biology, for we find man treated both as an animal and as a population problem.

The first chapter presents a detailed statistical study of five series of brain weights taken from different countries. Definite racial types are found. Brain weight correlates very poorly with body weight. The sexes are equally variable. Brain weight does not appear to be sensibly correlated with intellectual ability. These are only a few of the questions that are answered by the array of statistical data of this chapter. The next chapter tells us that there is very slight correlation between intelligence and the size of the head. Two methods for obtaining

correlation coefficients were used, the contingency method giving $0.14 \pm .04$ and the Pearson four-fold method $0.103 \pm .034$. This data would mean more to many readers if the long German statement of the material used (page 94) had been briefly summarized in English for the benefit of those who do not read German fluently.

The following two chapters on the sex ratio complete the first part of the book, which considers man as an animal. Data which show whether the children's parents are Italian, Spanish, Argentine, or combinations of these, from the vital statistics of Buenos Ayres, are used to show that the sex ratio, or the number of males per hundred females, is slightly higher from cross than from pure matings. Environmental and demographic conditions were analyzed, but failed to explain this difference in the sex ratio. Evidence is given to show that there is no relation between the time of fertilization and the sex ratio.

Part II is concerned with the biological aspects of vital statistics. Chapter five shows that if we measure variability of the two sexes by means of the congenital malformations which lead to the death of the individual, we find the female sex is slightly more variable than the male. This is shown by a study of the means, standard deviations and coefficients of variation. This chapter brings to the reader's attention the fact that vital statistics may be used to study evolution in the actual elimination of the unfit in the struggle for existence.

For deaths under one month and under one year the center of mortality occurs at 0.3 month and 0.3 year respectively. For the interval of under five years the center occurs at one year.

Next Pearl turns to the vitality of the people of America. This vitality is measured by the percentages of marriages producing children during 1919, the number of deaths, and the vital index or the ratio of the births to the deaths. He shows that there is a very considerable actual fusing going on between the foreign and native population. The vitality of the foreign born is greater than that of the native born. During 1919 the parents of 65% of the children born were native, of 10% one parent was native, and of only 25% were both parents foreign. Such data is more cheerful and helpful than the speculations of the sociologist. The trends of four vital indices,

$$\frac{100\ (\text{Births of whites of native parents})}{\text{deaths of native whites}},$$

$$\frac{100\ (\text{Births of whites both parents foreign})}{\text{deaths of foreign born whites}},$$

$$\frac{100\ (\text{Births of negroes})}{\text{deaths of negroes}},$$

and

$$\frac{100\ (\text{Births of whites})}{\text{deaths of whites}},$$

are presented using data from both rural and urban groups.

Following the treatment of vitality we find an analysis of the incidence of tuberculosis and how it is related to the constitution of the individual. Pearl seems to favor the hereditary factors rather than the infection factor though his study shows, however, that we must know more before we can justly reach any conclusion. Another chapter studies the longevity of the parents of the tuberculous and cancerous.

One of the most interesting chapters in the book is the one wherein various occupations of the British are classified into five groups based on the amount of physical activity involved. Then the mortality of each group at various ages was studied by Pearl in order to find out what relation, if any, existed between physical activity and mortality. After occupational accidents and hazards were excluded, he found that up to the age of about 40 there was no relation, but after age 40 a higher mortality was associated with the more strenuous activities. Outdoor occupations were found to be more healthful than indoor occupations. Pearl analyzed his data by means of differences in mortality after dividing the data into groups and the groups into quintiles.

Mortality tables are in existence for three animals only, man, the fruit fly and a rotifer. When these results are placed on a comparable scale, we find the highest mortality in the fruit fly, man is next and the rotifer has the least mortality. The curves are very similar.

Part III deals with public health and epidemiology. First Pearl presents an analysis of our food consumption in terms of primary food and secondary food. The secondary food includes those sources that are produced by the use of primary food, such as meat, etc. Of our food 47% was primary and 53% secondary. By studying the graphs for the different foods we discover, perhaps with surprise, that wheat furnished the greatest percentage of both our protein and carbohydrate. Also that we consume about 120 grams of protein a day when we actually need only 50 to 75 grams. Another chapter shows that the food conservation campaign during the war was really successful in that it cut down the amount of garbage produced in some 96 cities. The reviewer wonders if the price of food and its obtainability might not also have had some effect.

Before studying epidemics Pearl derives an index for the age distribution of a population. A very long chapter analyzes the influenza pandemic of 1918 showing how greatly it varied in different cities. Besides the duration of an epidemic, its standard deviation and number of peaks, the author suggests a number of indices for use in studying an epidemic.

The last part of the book is concerned with the population problem. The treatment starts by going back to Malthus and then pointing out that modern statistics show that population is outstripping food production though it is not following the particular mathematical expression of Malthus. Then the author stops to discuss "biology and the war," reviewing the causes of war as part of the inherent make-up of man. Pearl devotes several pages to the German military use of Darwinism to develop their war philosophy. He concludes from his statistical study of the war that a nation neither gains nor loses by war.

If one is interested in the stability of population, the data from England may be investigated. Here the ratio of 100 times the births divided by the deaths gradually increased from 1840 until the war with two exceptions due to influenza. During the war it dropped, but by 1920 it had risen to a higher place than the projection of the previous curve would indicate. The last two chapters of the book present Pearl's curve for population growth and its application to the growth of several countries. The first empirical curve of Pearl's suitable for this type of growth was a logarithmic parabola. While this curve gave good results, it was not in a general form. His general curve came from a study of the following five fundamental considerations: that a population must have a finite limit of area, an upper limiting asymptote, a lower limiting asymptote = 0, cycles of growth—successive ones being additive, and that the general shape of growth curves is that of an S. The general curve is of the form $x = k/(1 + me^{f(x)})$, where the $f(x) = a_1x + a_2x^2 + a_3x^3 \cdots$. This curve represents well the population growth in several countries and in a city. Methods for the actual fitting of the curve are given with the probable errors of the constants. For the growth of the United States $k = 197.27 \pm 0.55$, $m = 67.32 \pm 0.17$, and $a_1 = -0.0313 \pm 0.00013$. Pearl, after discussing the errors involved in the procedure, extrapolates the curve and predicts that we will reach the limiting value of our population of about 197 millions very shortly after 2100 A.D. The curves for population growth in 12 European countries and in Baltimore, Maryland, are given as well as that for the world. The limiting asymptote for the world population is given at approximately 2026 million.

Probably few books present as great a cross section through modern life and living conditions. For a group of collected papers it is quite coherent. This type of book is open to the objection that many of the chapters are so detailed as to detract from their interest to anyone not doing special work in the particular subject. Apparently the chapters are reprinted with a minimum of change from their original form of publication. One may, of course, skip that which does not interest him or is too detailed, so it is a moot point whether greater revision would have been desirable. The book should awaken greater interest in the field of vital statistics, both toward further analysis and toward making the reporting of them more comprehensive and accurate. With possibly a few exceptions the subjects treated are of great enough importance to merit the attention and careful consideration of all modern thinkers, and the evidence from this form of study should be welcomed and suitably evaluated. Notwithstanding the fact that much of the content of these studies is in the nature of a first approximation, they are truly a contribution to human biology which amply warrants their being brought together in this form. O. W. RICHARDS.

Medical Biometry and Statistics. By R. PEARL. Philadelphia, Saunders, 1923. 379 pages, illustrated. Price $5.00.

The mathematician will be interested and, perhaps, surprised to find a book devoted entirely to medical and biological statistics among the several new books

on this subject. Since the biological and medical sciences are fast becoming users of mathematics, it is unfortunate that more mathematicians are not interested in this new field which offers many problems of a new and more complex order than those of chemistry and physics.

This work on statistics is a new contribution to the subject, based on many years of teaching experience, and represents a novel method of teaching the subject. Raymond Pearl, Professor of Biometry and Vital Statistics in the School of Hygiene and Public Health at Johns Hopkins, has presented the subject with a minimum of formal mathematical proof, essentially for non-mathematical readers, but in such a logical manner that they can obtain an accurate knowledge of the limitations of the various methods and measures. Since the book is designed for the medical student, we find a considerable portion of the book devoted to vital statistics. To this extent it is specialized and does not represent the broader field of biological statistics. If the mathematician will keep this limitation in mind, he may turn to Pearl's book to see how rapidly this field is growing and for a further proof that mathematics is fast developing and is not the stationary subject that his college major students sometimes find in their texts.

The author first shows the need that the medical man has for statistical aid and then defines the subject and its general terms. Then he shows the student very briefly how vital statistics has grown from the first weekly bills of mortality in 1532 to its present status. This is followed by a short history of biometry. Chapter three presents the "raw data of biostatistics" according to the three ways of obtaining it, the census, registration, and the medical case record. This chapter is illustrated with census forms and records and the international list of the causes of death. The next chapter discusses, with many examples, the ways that such data may best be tabulated. The question of size of the class interval is well treated. A short chapter familiarizes the student with mechanical tabulation with regard to hospital records by use of Hollerith machines.

The sixth chapter contains an exceptionally fine treatment of graphical methods. Besides a complete discussion of all the common forms of graphs, we find methods for the preparation of nomograms. The need for arithlog graphs is indicated by the use of death rates. The following chapter teaches the student a proper conception of rates and ratios by showing him the differences between crude death rates and specific death rates and how the several vital indices are used. Should the reader desire to know just what a life table contains and something of the methods used in its preparation, he will find the answer in chapter eight. Then follows a chapter on "standardized and corrected death-rates" which indicates methods for determining the effect of age and sex on death as well as the relative health of the group considered.

At this point Pearl introduces the probable error concept showing how sampling may affect the results and also how it may be used as an aid in determining the reliability of the various measures used. This leads to an elementary consideration of probability, which is related to the life of the student by discussing the results of tossing a coin. The treatment is entirely limited to the

probability of experience that may be used in practical application. From this he leads to permutations and combinations, and then to the binomial theorem. The probability of concurrent events is briefly but adequately presented. From these concepts the author develops the normal curve as the limit of the point binomial when the probabilities are equal. Such development is perhaps suitable for the use made of the normal curve but a broader treatment of the binomial curve and its relation to the normal curve which can be developed by shifting the origin and by the use of Stirling's formula for large factorials would have been of use to the student of biometrics. The use of the normal curve and tests of sampling are presented with data from vital statistics.

The thirteenth chapter of the book takes up the methods of measuring variation such as the mean, mode, standard deviation, skewness, etc. It is interesting to note that while the standard deviation has appeared before, using this method of approach, these measures are left to almost the end of the book. The subject of correlation is treated with considerable completeness both for linear and non-linear regression. Another chapter contains an excellent brief discussion of partial correlation. Biological data are used for illustrating all of these methods. Several of the type curves of Pearson are illustrated by the variation of infant mortality in different population groups.

The last chapter is devoted to simple curve fitting by the method of least squares and is illustrated by fitting a straight line, a parabola, and a logarithmic curve to data representing the sitting heights of embryos.

The first appendix gives age and sex specific death rates. The second lists books and tables useful in the calculations; and the third contains an extensive list of formulæ from algebra, trigonometry, and calculus, as well as certain constants. The last two appendices contain a short table of the normal probability integral and sums of logarithms respectively. Bibliographic references are given at the end of each chapter and as occasional footnotes.

The book is well written and sufficiently non-technical to permit the advanced student or medical practitioner to use it profitably and with some appreciation of the logic back of the methods. Since most of the first part of the book pertains to vital statistics, and since most of the illustrative material is taken from that field, only parts of the book will interest the biology student. The book might have been made more useful to the biologist by the addition of methods for curve analysis and fitting other than least squares, and of a statistical discussion of problems of growth and of dynamic equilibria. The mathematician will find the book useful in preparing courses for students of public health and biostatistics, as well as those of medicine.

O. W. Richards.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB TOPICS.

FUNCTIONAL EQUATIONS.

By B. H. BROWN, Dartmouth College.

Cauchy [1] showed that the only real continuous solutions of the functional equations:

$$\begin{aligned} I \quad & \phi(x+y) = \phi(x) + \phi(y), \\ II \quad & \phi(x+y) = \phi(x) \cdot \phi(y), \\ III \quad & \phi(xy) \quad = \phi(x) + \phi(y), \\ IV \quad & \phi(xy) \quad = \phi(x) \cdot \phi(y); \end{aligned}$$

are ax, a^x, $a \log x$, x^a respectively.[2] These derivations are simple and are well adapted for a mathematical club talk.

Discontinuous solutions of these equations exist [3]; but the proof is neither simple nor suitable for our purposes. What is of very great interest is the totally depraved nature of these discontinuous solutions and the very slight restrictions which suffice to establish continuity. Thus a solution of Equation I is continuous under any one of the following restrictions:

(a) If $\phi(x)$ is continuous in the neighborhood of one value of x,

(b) if $\phi(x)$ is positive when x is positive,

(c) if there is any rectangle in the xy plane which does not contain a point of the locus $y = \phi(x)$.

Finally [4] a fifth functional equation

$$V \quad \phi(x+y) + \phi(x-y) = 2\phi(x) \cdot \phi(y)$$

has for continuous solutions: $\cos ax$; $(A^x + A^{-x})/2$.

Application A. Equation V is of importance in statics in establishing the theorem of the parallelogram of forces.[5] The same theorem is established by Darboux [6] by the use of Equation I; the statical hypothesis which suffices for continuity is worthy of attention.

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[1] *Cours d'Analyse*, Paris, 1821, pp. 103ff.; for a more recent treatment *cf.* Tannery, *Théorie des fonctions d'une variable*, sec. ed., Paris, 1904, p. 275.

[2] There are, of course, the trivial particular solutions $\varphi(x) \equiv 0$ for Equations II and IV.

[3] Hamel, *Math. Ann.*, vol. 60, 1905, p. 459; Schimmack, *Diss.* Halle, 1908.

[4] Other functional equations arising from problems in permutations and combinations, applying to integral values of the argument may be found in treatises on algebra. *Cf.* C. Smith, *A treatise on algebra*, London, 1888, pp. 286, 287.

[5] Voss, *Enc. der Math. Wiss.*, IV, 1, p. 66. *Cf.* d'Alembert, Paris, *Mem. de l'Acad.*, 1769, p. 278.

[6] *Bull. sci. math.*, vol. 9, 1875, p. 281.

Application B. Equation V serves as a definition of the circular functions, and shows the close relationship between these functions, spherical trigonometry, the plane Euclidean and non-Euclidean trigonometries, and statics.[1]

Application C. THEOREM: *If there exists a correspondence between two series of elements which preserves harmonic ratio, the correspondence is projective.* This theorem, without any assumption of continuity for the correspondence, is established [2] by the aid of Equation I and restriction (b) which is a consequence of our hypothesis.

Application D. Equation II has been used to establish Laguerre's projective definition of angle.[3] If OP and OQ are the minimal lines through the vertex of the angle $\theta = AOB$, Laguerre showed that

$$\theta = (\log \{AB; PQ\})/2i,$$

where $\{AB; PQ\}$ is the cross-ratio of the four lines $OA, OB; OP, OQ$. To prove this [4] we first show that the cross-ratio is independent of translation and rotation and hence is actually some function $f(\theta)$ of the angle θ. The well-known identity

$$\{AB; PQ\} = \{AC; PQ\} \cdot \{CB; PQ\}$$

is then equivalent to $f(\theta_1 + \theta_2) = f(\theta_1) \cdot f(\theta_2)$, or Equation II.
Hence

$$f(\theta) = a^{\theta},$$

$$\theta = a' \log \{AB; PQ\},$$

and a' may be shown to be $1/2i$ for the special case $AOB = \pi/2$.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF COOPER UNION, New York City.
[*1924*, 496.]

The officers for the year 1924–1925 were: President, Samuel Lubkin '27; Vice-president, Fred Miller '26; Secretary-Treasurer, Sydney Helprin '27. Meetings were held at intervals of three weeks throughout the year. The slide-rule presented by the Club to the member of the first year class having the highest standing in mathematics was won by Peter Douglas '28.

Following is the program for the year:

October 28, 1924: "Mathematics in industrial research"—Thornton C. Fry of the Western Electric Company.

November 18, 1924: "Introduction to the solution of modern geometric problems"—Aaron Rabinowitz '27.

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[1] *Cf.* Andrade, *Bull. Soc. math.*, vol. 28, 1900, p. 58.

[2] Darboux, *Math. Ann.*, vol. 17, 1880, p. 54.

[3] Laguerre, *Nouv. Ann.*, (1) vol. 12, 1853, p. 64; *Oeuvres* 2, Paris, 1905, pp. 12–13; but see also *Enc. der Math. Wiss.*, III AB9, p. 901 where Laguerre's result as a *definition* is credited to Cayley.

[4] The spirit if not the letter of this method is due to Klein, *Math. Ann.*, vol. 4, 1871, p. 584. In the same spirit Equations I, II, and V are frequently employed in connection with angle and distance in the non-Euclidean geometries.

December 9, 1924: "Notions about vectors"—Thaddeus Slonczewski '26.
January 6, 1925: "Some original methods of solving geometrical problems"—Samuel Lubkin '27.
January 27, 1925: "The mathematics of linkages, with an exhibition of models"—Samuel Lubkin '27.
February 17, 1925: "Some results in the theory of numbers"—Fred Miller '26.
March 10, 1925: "When is a check not a check?"—Sydney Helprin '27.
March 31, 1925: "History of the development of the slide-rule"—Edwin Schwarz '27.
April 21, 1925: "Crinkly curves"—Professor H. W. Reddick. Election of officers.

(Report by Sydney Helprin, Secretary.)

Mathematical Club, New Jersey College for Women, New Brunswick, N. J.

[*1924*, 352.]

At the regular monthly meetings of the Mathematical Club at the New Jersey College for Women, Rutgers University, the following papers were read:
"Life and work of Archimedes" by Elizabeth Sinton.
"Life of George Boole" by Janey Kudlich.
"The Solution of some problems" by Dorothy McFarland.
"Invariants and covariants"
"Problems involving complex numbers" } by Professor Richard Morris.
"The discriminant"
"History of complex numbers" by Katherine Mitchell.
"A geometrical fallacy" by Marjorie Cadwallader.
"Problems involving the law of cosines" by Professor Starke.
"Mathematics in the industries" by Dorothy Brown.
"Taylor's theorem for 2 and 3 variables" by Vera Joslin.
"Euler's theorem for 3 or more variables" by Myrtle Hughes.
"Life of John Landen" by Ada Salmon.

(Report by Professor Morris.)

Mathematical Club, Rutgers University, New Brunswick, N. J.

[*1924*, 352.]

The officers of the Men's Mathematical Club of Rutgers University for the past year were: President, R. M. Walter; Vice-president, L. J. Paradiso; Secretary-Treasurer, R. J. Seeger.

The following papers were read at the regular meetings during the year:
"Complex numbers" by Professor Brasefield.
"Time solids" by W. H. Mitchell, Jr.
"Method of least squares" by Harry Rolnick.
"Applications of invariants and covariants" by Professor Morris.
"Orthogonal transformations" by L. J. Paradiso.
"Theorems of Pascal and Brianchon" by W. H. Mitchell, Jr.
"Hyperbolic functions" by R. M. Walter.
Statement of the problem of three bodies by R. J. Seeger.
Transformation of the general elliptic integral by L. J. Paradiso.
Dimensional units by Harry Rolnick.

(Report by Professor Morris.)

White Mathematics Club, University of Kentucky, Lexington, Ky.

[*1924*, 452.]

Oct. 10, 1924. Election of the following officers: President, Professor J. M. Davis; Secretary, Professor H. H. Downing. It was decided to make Eddington's book on *Einstein's theory of relativity* the basis for the year's club work and to have joint meetings with the physics club to consider the problem from a physical as well as a mathematical point of view.
Oct. 16. "Charts of mathematical history" by Professor E. L. Rees.
Oct. 30. "What is geometry? Relativity" by Dr. P. P. Boyd.

Nov. 11. "Demonstration of the Michelson-Morley light experiment" by Professor W. S. Webb, physics department. "Discussion of the Michelson-Morley light experiment" by Dr. Otto Koppius, physics department.
Nov. 18. "Lorentz's equations in the theory of relativity" by Dr. M. N. States, physics department.
Dec. 18. "Applications of the Lorentz equations to certain problems" by Professor J. M. Davis.
Jan. 15, 1925. "Time, the fourth dimenson" by Professor E. L. Rees.
Feb. 5. "Fields of force" by Professor H. H. Downing.
Feb. 19. "Kinds of space" by Dr. F. Elizabeth Le Stourgeon.
Feb. 26. "Postulates for the Lorentz transformations" by Mr. J. C. Nixon, instructor.
Mar. 5. "The Bucherer experiment" by L. A. Pardue, '25.
Mar. 11. "The mathematics of Einstein's law of gravitation" by Professor E. L. Rees.
Mar. 16. "Geodesics in general relativity" by Professor H. H. Downing.
Apr. 16. "Mathematical wrinkles" by all present.
May 7. "Properties of congruences" by Mr. D. E. South, instr.
May 14. "Vector treatment of the motion of a rigid body in a plane" by Mr. T. Andrew, instr. "Algebraic solution of quartic equations" by R. S. Park, '25.
May 21. "Some mathematicians who have contributed to the theory of equations" by Grace Richards, '25. "Proof of Pascal's theorem" by M. C. Brown, gr.
May 27. "Algebraic equations" by Mary H. Cooper, '25. "Determinants" by Eva Weller, '25.

(Report by Professor H. H. Downing.)

The Mathematics Club of the University of Kansas, Lawrence, Kansas.

[*1921*, 88.]

The officers of the Mathematics Club of the University of Kansas for the year 1924–25 were: President, Mildred Woodside '25; Vice-president, Maude Long '25; Secretary-Treasurer, Violet Shoemaker '25; Faculty Adviser, H. E. Jordan.

The following topics were presented at the meetings:
October 6, 1924: "The greatest word in mathematics" by Professor G. W. Smith.
October 20: "Railroad curves" by Professor F. A. Russell.
November 3: "Various proofs of the Pythagorean theorem" by Violet Shoemaker '25.
November 17: "The slide rule" by Professor H. E. Jordan.
December 1: "Magic squares" by Lucille Heil '25.
December 15: "Graphical solution of cubic equations" by Forrest Noll '25.
January 12, 1925: "Newton and Leibniz" by Maude Long '25; "Properties of the catenary" by Mildred Woodside '25.
February 23: "Japanese mathematics" by Leta Galpin '25.
March 5: "Schuster's periodogram" by Professor Dinsmore Alter.
March 16: "Solution of equations by the iteration process" by Elizabeth Bolinger '26.
April 6: "A new method of extracting roots of numbers" by C. A. Reagan, Gr.
April 20: "Interpolation formulae" by Wesley M. Roberds '25.
May 4: "Normals to a parabola" by Lester Lehnberg '25.
May 18: Annual picnic.

(Report by Miss Shoemaker.)

Goucher College Mathematics Club, Baltimore, Maryland.

[*1925*, 91.]

Nov. 6. "The trisection of an angle."
History of the problem; solution by the hyperbola and quadratrix, by Sara Scott '27.
Proof of the incorrectness of a proposed construction, by Anna Grimm '26.
Analytic proof that trisection by ruler and compass is impossible, by Elizabeth Bauernschmidt '25 and Louise Kinnamon '25.
Nov. 20. "Numerical calculation."
Calculating prodigies, by Amelia Frank '27.

History and construction of adding machines, by Margaret Heinzerling '26 and Gwendolyn Eichhorn '26.

Tricks with numbers by Ruth Farr '25.

Dec. 18. "The theory of relativity" by Professor F. P. Lewis.

Feb. 12. "Typical theorems of pure projective geometry" by Catherine Snyder '25.

"The slide rule" by Marvell Weinberg '27 and Lillian Peper '27.

Report on the December meetings of the Mathematical Association in Washington, Professor Bacon and Professor Lewis.

March 12. "Mathematics in sculpture and painting" by Professor Hans Froelicher, Department of Art, Goucher College.

April 16. "The nine-point circle and Feuerbach's theorem" by Leah Seidman '26.

"A ruler and compass construction for the regular pentagon and decagon" by Doris Heineman '27.

May 21. Annual club picnic.

(Report by Professor Lewis.)

Mathematics Club of the State University of Iowa, Iowa City, Iowa. [*1924*, 496.]

The following officers were elected for the year 1924–1925: President, Howard K. Hughes, G; Secretary-Treasurer, Arthur H. Blue, G; Faculty Adviser, Professor E. W. Chittenden. The attendance has been about thirty at each of the nine meetings. A membership fee of twenty-five cents has amply provided for tea and cakes, served before each program. The following topics were presented at the meetings:

"Pencils of conics" by H. K. Hughes, G.
"Duhamel's theorem" by R. J. Hannelly, G.
"An octaval number system" by Frances E. Baker, G.
"Accuracy of interpolation" by A. H. Blue, G.
"Applications of determinants to geometry" by Eva M. Schillig, G.
"Hyperbolic functions" by W. J. Davidson, '25.
"The steel square" by W. L. Hunter.
"Quadrature formulas" by J. Van S. Longenecker, G.
"The incomplete gamma function" by Clair Kirkpatrick, G.
"What are the stars?" by Dr. D. H. Menzel.

(Report by Mr. Blue.)

Mathematics Club of the University of Arkansas, Fayetteville, Arkansas.

During 1924–25 the following topics were treated:
"Some elementary number theory" by F. E. Taylor.
"Continued fractions" by Jewell Hughes.
"Modern geometry" by A. D. Campbell.
"Mathematical recreations" by T. T. Spitzberg.
"History of geometry" by Emily Heston.
"Determinants" by C. T. Willis.
"Relation of mathematics to physics" by S. R. Parsons.
"Theory of probability" T. L. Edmiston.
"A broader idea of integers" by Jewell Hughes.
"The fourth dimension" by A. Campbell.
"Conic sections" by F. E. Taylor.
"The Game of Nim" by Grace Harrison.

(Report by Professor Campbell.)

Simpson Mathematical Society, University of Florida, Gainesville, Florida.

The Simpson Mathematical Society of the University of Florida was organized this year. All interested in mathematics are eligible to membership. The officers of the Society for the spring of 1925 were as follows:

Z. M. Pirenian, President; V. C. Steen, Vice-president; S. H. Huffman, Secretary; W. Stanwix-Hay, Reporter.

The following topics have been discussed in the regular weekly meetings:

March 10. The equation $e^{i\pi} = \cos\pi + i\sin\pi = -1$.
March 18. Mathematical induction; Euclid's fifth postulate.
March 25. Cardan's solution of the cubic; Mathematical puzzles.
April 1. The teaching of mathematics; Mathematical tricks.
April 8. Magic squares; Complex numbers.
April 15. Theory of investment problem; Solution of the quartic.
April 22. The fourth dimension; Hyperbolic functions.
April 29. Einstein theory; Mathematical puzzles.
May 6. Theory of probability; Mathematics of biology.
May 13. Life of Leibniz; The theory of iteration.
May 20. Logarithms; The slide rule.

(Report by S. H. Huffman, Secretary.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3144. Proposed by H. S. CARSLAW, The University, Sydney, Australia.

Let

$$y = \frac{1}{n}\sum_{1}^{n}(n-r+1)\frac{\sin(2r-1)x}{2r-1} = S_n(x). \qquad (0 \leqq x \leqq \tfrac{1}{2}\pi)$$

For a given positive integer n, it is easy to show that the maxima occur when x is $\frac{\pi}{n+1}$, $\frac{2\pi}{n+1}$, $\frac{3\pi}{n+1}$, $\cdots$, the last being at $\frac{1}{2}\pi$, if n is odd, and at $\frac{1}{2}\pi - \frac{1}{2(n+1)}\pi$, if n is even. Also the minima occur when x is $\frac{\pi}{n}$, $\frac{2\pi}{n}$, $\frac{3\pi}{n}$, $\cdots$, the last being at $\frac{1}{2}\pi$, when n is even, and at $\frac{1}{2}\pi - \frac{1}{2n}\pi$, when n is odd.

It is required to show that, for a given n, the ordinates of the maxima continually increase as x increases from 0 to $\frac{1}{2}\pi$, and similarly for the minima. Also that the ordinate of the last maximum continually diminishes as n increases, and that its limit is $(\pi/4)$ when $n \to \infty$.

3145. Proposed by W. D. CAIRNS, Oberlin College.

The center of gravity of any zone of a certain surface of revolution lies midway between the bases of the zone. What is the surface?

3146. Proposed by W. C. EELLS, Whitman College.

For what integral values of s and n is it true that the nth power of any integer leaves the same remainder when divided by s as does the integer itself when divided by s? (Generalized from Problem 873, *School Science and Mathematics*, March, 1925.)

3147. Proposed by J. A. BULLARD, U. S. Naval Academy.

The cylinder $(x/a)^2 + (y/b)^2 = 1$, passes through the sphere, $x^2 + y^2 + z^2 = a^2$. If e is the eccentricity of the ellipse $(x/a)^2 + (y/b)^2 = 1$, show that

(a) the surface cut from the sphere $= 8a^2 \arccos e$,

(b) the volume cut from the sphere $= 8a^3[e^2\sqrt{1 - e^2} + \arccos e]/3$.

2677 [1918, 75]. Proposed by R. K. MORLEY, Worcester, Mass.

A quarter-mile track is to be constructed, having semi-circular ends and straightaway sides. It is required to have the rectangular part of enclosed field referred to in No. 12, Granville's *Calculus*, page 116, as large as possible. Find length of the straightaways.

Also, required to inscribe the maximum rectangle in a track of length l, with semi-circular ends and straightaway sides, assuming that two sides of the rectangle are parallel to the straightaways. Find the length of the straightaways and the dimensions of the rectangle.

2762 [1919, 70]. Proposed by N. P. PANDYA, Amreli, India.

$ABCD$ is a cyclic quadrilateral inscribed in an ellipse. $AB = 2BC$ and $CD = 2DA$. Find the eccentricity of the ellipse in terms of the sides of the quadrilateral.

2769 [1919, 171]. Proposed by B. J. BROWN, Kansas City, Mo.

Expand in powers of x as far as x^2 the function $\dfrac{\cosh \lambda x}{\cosh \lambda} - x\dfrac{\sinh \lambda x}{\sinh \lambda}$ in which λ is a positive constant.

Prove that, if $\lambda \tanh \lambda > 2$, the function has only one maximum value for $x > 0$ and that the value of x for which the maximum occurs is less than 1. (India Civil Service. 1912.)

2770 [1919, 171]. Proposed by A. M. HARDING, University of Arkansas.

Solve the differential equation

$$\frac{d^3x}{dt^3} - 2(\mu t + \lambda)\frac{dx}{dt} + (1 - \mu)x = 0,$$

where λ and μ are constants.

SOLUTIONS.

2874 [1921, 37]. Proposed by J. L. RILEY, Stephenville, Texas.

Show that the equation,

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + k = 0,$$

has some imaginary roots if $a^2 - 2b < n\sqrt[n]{k^2}$; $a, b, \cdots, k$ are supposed real.

Note. This result is a particular case of a theorem contained in a paper published a few years ago.—EDITORS.

SOLUTION BY OTTO DUNKEL, Washington University.

A proof of the theorem mentioned in the note above may be found in the paper by Dunkel on "Generalized Geometric Means and Algebraic Equations" in *Annals of Mathematics*, 2d ser., vol. 11, no. 1, October 1909, pp. 24–25.

3103 [1924, 498]. Proposed by OTTO DUNKEL, Washington University.

Given the equation

$$y = \sum_{i=1}^{i=\infty} \frac{(i+1)^{i-1}}{i!}x^i, \qquad |x| < 1/e,$$

find x in terms of y in finite form.

SOLUTION BY THE PROPOSER.

It will be more convenient to write

$$y = \sum_{t=0}^{t=\infty} \frac{(i+1)^{i-1}}{i!} x^i. \tag{1}$$

This change in the lower limit does not alter the problem in any essential respect.

The following identity in a will be proved which will be used later:

$$\begin{aligned}(t+a+1)^t &= \sum_{t=0}^{t=t} (a+i)^i (t-i+1)^{t-i-1} {}_tC_i, \left({}_tC_i = \frac{t!}{i!(t-i)!} \right), \\ &= \sum_{t=0}^{t=t} (a+t-i)^{t-i} (i+1)^{t-i} {}_tC_i.\end{aligned} \tag{2}$$

PROOF. In the second form of (2) set $(a+t-i)^{t-i} = [(a+t+1)-(i+1)]^{t-i}$, and develop this last expression by the Binomial Theorem. The right side of (2) then becomes the double sum

$$\sum_{t=0}^{t=t} \sum_{j=0}^{j=t-t} (a+t+1)^j (i+1)^{t-j-i} {}_tC_i \; {}_{t-i}C_j (-1)^{t-i-j}.$$

Since ${}_tC_i \; {}_{t-i}C_j = {}_tC_j \; {}_{t-j}C_i$, we may write the above double sum in the form

$$\sum_{j=0}^{j=t} (a+t+1)^j \; {}_tC_j (-1)^{t-j} \sum_{i=0}^{i=t-j} (i+1)^{t-j-i} \; {}_{t-j}C_i (-1)^i.$$

Now the last single sum (with respect to i) is zero, if $j \neq t$ (see the solution of **3108 [1925, 388]**); but, if $j = t$ and hence $i = 0$, it is equal to unity. Thus the double sum reduces to the single term $(a+t+1)^t$ and this is what we wished to prove. The first form of the right member of (2) is obtained from the second form by replacing i by $t-i$, and is therefore equal to the second form.

Turning now to (1) we readily verify for the range of values given for x that

$$xy' + y = \sum_{t=0}^{t=\infty} \frac{(i+1)^i}{i!} x^i.$$

Hence

$$\begin{aligned}y(xy'+y) &= \left(\sum_{t=0}^{t=\infty} \frac{(i+1)^{i-1}}{i!} x^i \right) \left(\sum_{t=0}^{t=\infty} \frac{(t+1)^t}{t!} x^t \right), \\ &= \sum_{t=0}^{t=\infty} \frac{x^t}{t!} \sum_{i=0}^{i=t} (i+1)^{i-1} (t-i+1)^{t-i} {}_tC_i.\end{aligned} \tag{3}$$

Setting now in the identity above $a = 1$, we may replace the second single sum in the double summation by $(t+2)^t$. We have then

$$y(xy'+y) = y' \qquad \text{or} \qquad y\,d(xy) = dy. \tag{4}$$

It now follows that $xy = \log y + A$. In this case $A = 0$, since in (1) if $x = 0$, $y = 1$. We then find the final result

$$x = \frac{\log y}{y} \qquad \text{or} \qquad e^{xy} = y. \tag{5}$$

REMARKS. The second equation in (5) occurs in the solution of **3071 [1925, 139]** with the notation $e^{mh} = m$. Here h is given, and hence (1) gives that solution of the equation which was used in the solution of the problem but which was not written in this explicit form. The analysis in the solution of 3071 with certain additions may be used to give another derivation of the result above. We have thus found the values of the coefficients a_i which occurred in that solution,

$a_i = (i+1)^{i-1}$. The equation (6) of that solution may now be written

$$m = \sum_{i=0}^{i=\infty} \frac{(i+1)^{i-1}}{i!} h^i = \frac{\sum_{i=0}^{i=\infty} \frac{[x+(i+1)h]^i}{i!}}{\sum_{i=0}^{i=\infty} \frac{[x+ih]^i}{i!}} \tag{6}$$

where x may be any number.

It may be of interest to write a second identity which may be proved in a similar manner,

$$\frac{(t+a+1)^t}{a} = \sum_{i=0}^{i=t} (a+i)^{i-1}(t-i+1)^{t-i}\,{}_tC_i. \tag{7}$$

This identity could have been used in place of (1), after setting $a = 1$, but (1) was chosen as corresponding in its first form to (4) [**1925, 139**].

3109 [1925, 46]. Proposed by the late J. W .NICHOLSON.

Find two rational numbers which separate the roots of the equation $x^3 - ax^2 + bx - c = 0$.

SOLUTION[1] BY A. A. BENNETT, University of Texas.

Given the cubic $f(x) = x^3 - ax^2 + bx - c$, we have on differentiating $f'(x) = 3x^2 - 2ax + b$, $f''(x) = 6x - 2a$. The discriminant of f' (according to one definition) may be written as $9l = a^2 - 3b$, and the value of $f(x)$ for $f''(x) = 0$ may be denoted by $q = -2a^3/27 + ab/3 - c$. We may remark that by the translation $u = x - (a/3)$, $f(x)$ may be put in the reduced form $u^3 - 3lu + q$. The discriminant of the cubic is $\Delta = 27(4l^3 - q^2)$. The problem has a meaning only if the roots are real and distinct and therefore only if Δ, and hence also l, is positive.

If l is the square of a rational number, an explicit solution is always afforded by the pair $\frac{1}{3}a + \sqrt{l}$, $\frac{1}{3}a - \sqrt{l}$, since these are the roots of $f'(x) = 0$, which by Rolle's theorem separate the roots of the original equation. If $\sqrt{l}$ is irrational, rational numbers can always be found sufficiently close to the roots of $f'(x) = 0$, so as to still lie in the given intervals. In particular if $q = 0$, $27f(x)$ factors into $(3x - a)[(3x - a)^2 - 27l]$. Thus if r is any nonvanishing rational number for which $r^2 < 3l$, we shall have as a solution the pair of numbers $\frac{1}{3}a + r$ and $\frac{1}{3}a - r$.

We are interested however in an explicit rational formula for the general case, namely for $q \neq 0$ and l not a perfect square. The following pair is in fact such a solution, $\frac{1}{3}a$, $(ab - 9c)/(2a^2 - 6b)$. The latter expression may also be written as $\frac{1}{3}a + q/(2l)$. Indeed $f(\frac{1}{3}a) = q$, and $f(\frac{1}{3}a + q/(2l)) = -q\Delta/(6l)^3$, which, since Δ and l are positive, is of opposite sign to q. To complete the proof, we shall consider separately the two cases of $q > 0$ and of $q < 0$. For $q > 0$, we have $-\infty < \frac{1}{3}a < \frac{1}{3}a + q/(2l) < +\infty$, and $f(-\infty) < 0$, $f(\frac{1}{3}a) > 0$, $f(\frac{1}{3}a + q/(2l)) < 0$, $f(+\infty) > 0$. On the other hand, for $q < 0$, we have $-\infty < \frac{1}{3}a + q/(2l) < \frac{1}{3}a < +\infty$, and $f(-\infty) < 0$, $f(\frac{1}{3}a + q/(2l)) > 0$, $f(\frac{1}{3}a) < 0$, $f(+\infty) > 0$. Hence the two given values separate the roots in either case. The proof is also obvious geometrically. One has merely to compare the points of intersection with the X-axis of the vertical line through the point of inflexion, and of the straight line which joins the point of inflexion with both of the bend-points.

3110 [1925, 46]. Proposed by J. L. RILEY, Stephenville, Texas.

If the curves $a_1x^m + b_1y^m + c_1 = 0$ and $a_2x^m + b_2y^m + c_2 = 0$ touch at a point, find the ratio of their radii of curvature at that point.

NOTE BY NORMAN ANNING, University of Michigan.

On page 280 of Loria, *Spezielle alg. u. transz. ebene Kurven*, the following theorem is proved. "Wenn zwei auf dieselben Axen bezogene Lamé'schen Kurven mit den Indices m und m' sich in einem Punkte berühren, so wird das Verhältnis der Krümmungen in diesem Punkte durch $(m-1)/(m'-1)$ ausgedrückt." In this problem, $m = m'$ and the required ratio has the value 1.

Also solved by J. A. BULLARD, ALICE A. GRANT, HARRY LANGMAN and H. A. SIMMONS.

[1] A solution of this problem by C. F. Gummer has already been published (*1925*, 389). EDITOR.

3113 [1925, 94]. Proposed by A. S. Wiener, Cornell University.

A man is paying off a mortgage of N dollars, interest i per cent. annually, by monthly installments of a dollars (where $a > iN/1200$, the first month's interest). How many months will it take him to pay off the mortgage?

SOLUTION BY C. C. WYLIE, University of Iowa.

It will be assumed that the first month's interest is due at the end of the first month when the first payment is made. If $r = 1 + i/1200$, the N dollars at the end of n months will amount to Nr^n, while the n payments of a dollars each will amount to

$$a(1 + r + r^2 + \cdots + r^{n-1}) = a(r^n - 1)/(r - 1).$$

Setting these two equal and solving for n we have

$$n = -\frac{\log [1 - N(r-1)/a]}{\log r}.$$

If $N = 100$, $i = 6$, $a = 1$, then $n = 139$ months approximately.

Also solved by W. S. BARLOW, J. A. BULLARD, L. A. EASTBURN, J. M. EARL, PHILIP FITCH, ALICE A. GRANT, MICHAEL GOLDBERG, ELMER LATSHAW, HARRY LANGMAN, C. H. LEHMANN, G. A. LYLE, and A. W. RICHARDSON.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. KUHN, Ohio State University, Columbus, Ohio.

At the University of Washington, Associate Professor R. M. WINGER has been promoted to a professorship.

Mr. H. B. LEMON of Denison University and Mr. W. O. MENGE have been appointed instructors at the University of Michigan.

Professor ANNA J. PELL of Bryn Mawr was married on July 6, 1925, to Professor A. L. WHEELER of Princeton University. Mrs. Wheeler has been made head of the mathematics department at Bryn Mawr to succeed Professor CHARLOTTE A. SCOTT.

Assistant Professor A. L. NELSON of the University of Michigan has been appointed professor of mathematics at the Detroit Junior College.

Professor T. P. NUNN of the University of London gave a number of courses at the summer session of Columbia University. Afterward he visited Cornell, Purdue, University of Chicago, and University of Michigan.

Professor B. M. WALKER has recently been made President of the Mississippi Agricultural and Mechanical College. He was president of the new Mississippi-Louisiana Section of the Association during its first year just closed.

Miss EMMA L. KONANTZ, for a number of years connected with the department of mathematics of Ohio Wesleyan University, but recently on the faculty

of Peking University, Peking, China, has been advanced to a full professorship in mathematics in that institution.

Dr. D. L. HOLL, for the last two years assistant professor of mathematics at Ohio Wesleyan University, has accepted a like position at Iowa State College. Mr. R. L. NEWLIN, formerly of Guilford College, N. C., has been appointed to succeed Dr. Holl at Ohio Wesleyan.

At the University of Arkansas, Assistant Professor A. D. CAMPBELL has been promoted to an associate professorship of mathematics.

Professor L. M. HOSKINS, of the department of applied mathematics at Stanford University, has retired from active teaching.

Dr. H. B. CURTIS, of Northwestern University, has been appointed associate professor of mathematics at Marquette University.

Dr. L. L. SMAIL, of the University of Oregon, and Mr. H. S. VANDIVER, of Cornell University, have been appointed associate professors of mathematics at the University of Texas.

Professor MAX MASON, of the department of mathematics at the University of Wisconsin, has been elected president of the University of Chicago.

The death is announced of Professor B. H. KERSTEIN, of the State Teachers College, Silver City, New Mexico.

Professor W. F. SHENTON, of the United States Naval Academy, has been appointed professor of mathematics and physics at the American University, Washington, D. C.

Professor H. L. SMITH, of the University of the Philippines, has been appointed to an assistant professorship at the University of Minnesota.

At the University of Pennsylvania the following promotions and new appointments are announced in the mathematics department: Dr. M. J. BABB from assistant professor to professor, Dr. J. D. ESHLEMAN from instructor to assistant professor, and Mr. N. E. RUTT from assistant to instructor; Dr. P. A. CARIS, assistant professor; Mr. W. A. BRISTOL, instructor; Mr. H. M. LUFKIN, instructor; and Mr. M. BROOKS and Mr. L. ZIPPIN, assistants.

At the University of Iowa, Associate Professor E. W. CHITTENDEN has been promoted to a full professorship. Dr. ROSCOE WOODS has been promoted to an assistant professorship.

Dr. D. H. MENZEL, University of Iowa, has been appointed assistant professor of astronomy at Ohio State University.

Dr. L. E. WARD of Harvard University has been appointed instructor in mathematics at the University of Iowa.

Mr. E. E. ERICKSON of the University of Iowa has been appointed assistant professor in Miami University.

Mr. H. K. HUGHES, University of Iowa, has been appointed instructor in mathematics, University of Kansas.

Mr. D. O. STREYFFELER of the University of Iowa has been appointed instructor in mathematics, University of Kentucky.

Dr. ISRAEL MAIZLISH has been appointed associate professor of physics at Centenary College, Shreveport, La.

Miss DORA E. KEARNEY has been appointed professor of mathematics at Iowa State Teachers College, Cedar Rapids.

Assistant Professor F. M. WEIDA, of Montana State College, has been appointed assistant professor of mathematics at Lehigh University.

The following appointments to instructorships are announced:

University of Illinois, Dr. R. M. MATHEWS.

Northwestern University, Mr. O. E. BROWN.

College of the City of Detroit, Mr. WILLIAM BORGMAN and Miss LUCILLE CHALMERS.

Assistant Professor J. W. MILLER, of the University of Pennsylvania, died Sept. 11, 1925.

On September 11, 1925, Prof. ERNST VON HAMMER of the Technische Hochschule of Stuttgart, Germany, died at the age of 67 years. For some forty years he had filled the chair of geodesy and practical astronomy in that institution. He was the author of numerous books and articles.

The University of Strasbourg, in addition to the standard courses on analysis, astronomy, and mechanics, is offering this year the following advanced courses which are particularly fitted to prepare candidates for the "diplôme d'études supérieures de Mathématiques" and for the Doctorates:

First Semester (November, 1925, to February, 1926)—By Professor BAUER, Lorentz theory, Theory of quanta; by Professor CERF, Outer multiplication and derivation; by Professor FRÉCHET, Integration of functionals, Introduction to nomography.

Second Semester (March 1 to June 15)—By Professor BAUER, Spectra and dynamics of atoms; by Professor FRÉCHET, Integration of functionals, Advanced nomography; By Professor VALIRON, Normal families of meromorphic functions; by Professor VILLAT, Conformal representation of minimal surfaces; by Professor THIRY, Advanced analytic geometry, Remarkable curves and surfaces.

THE CHAUVENET PRIZE FOR MATHEMATICAL EXPOSITION.

In March 1925, PRESIDENT COOLIDGE proposed that the Association establish a prize for special merit in mathematical exposition. The proposition was sanctioned by mail vote of the Trustees and a committee consisting of Professors A. J. KEMPNER, Chairman, LOUISE D. CUMMINGS and D. R. CURTISS was

appointed to formulate the details. This committee presented a report at the Ithaca meeting which in somewhat modified form was adopted by the Trustees. The substance of the report is as follows:

The committee believe that the proposed prize will exert a desirable influence on the production of high-grade exposition articles. They adopted the name suggested by President Coolidge, namely, "The Chauvenet Prize for Mathematical Exposition." For a study of the life and influence of WILLIAM CHAUVENET, 1820–1870, Professor of mathematics in the U. S. Navy, 1847–1859, President of the Academic Board of the Navy, 1847–1850, Professor of mathematics and natural philosophy at Washington University, St. Louis, Mo., 1859–1869, Author of many works and treatises, they refer to an article by Professor W. H. ROEVER, in *Washington University Studies*, Vol. XII, *Scientific series*, No. 2, 1925.

The Chauvenet Prize is to be awarded every five years for the best article of an expository character dealing with some mathematical topic, written by a member of the Mathematical Association of America and published in English in a journal during the five calendar years preceding the award. This prize will not be awarded for books, even though a large portion of mathematical books are mainly or completely expository in character, such as textbooks. They bring their own reward in the form of royalties.

The amount of the prize was fixed at one hundred dollars, an amount which the committee deemed sufficiently large to be attractive apart from the honor of the award. The cash for the first award has been provided by a friend of the Association. Thereafter, it will be supplied from the Association treasury, one fifth of the amount being set aside each year for the purpose.

The first award is to be made at the annual meeting in December 1925, covering the five-year period ending with the calendar year 1924.

It is provided that the award shall be determined at each quinquennial period by a scrutinizing committee of three to be appointed by the president of the Association and that this committee should be restricted as little as possible, aside from the specifications mentioned in the foregoing paragraphs. President Coolidge appointed the scrutinizing committee for the first award as follows:

E. B. VAN VLECK, Wisconsin, *Chairman*,
ANNA J. P. WHEELER, Bryn Mawr,
W. C. GRAUSTEIN, Harvard.

THE MAY MEETING OF THE IOWA SECTION.

The fourteenth regular meeting of the Iowa Section of the Mathematical Association of America was held at Iowa State Teachers College, Cedar Falls, Iowa, on May 1 and 2, 1925, in affiliation with the annual meeting of the Iowa Academy of Science.

Among those attending were the following twenty-four members of the Association: Julia T. Colpitts, I. S. Condit, Marian E. Daniells, R. D. Daugherty, R. M. Deming, C. W. Emmons, Annie W. Fleming, C. Gouwens, H. K. Hughes, Daniel Kreth, F. M. McGaw, J. V. McKelvey, Martha McD. McKelvey, E. A. Pattengill, J. F. Reilly, Fred Reusser, H. L. Rietz, Maria M. Roberts, E. R. Smith, G. W. Snedecor, J. S. Turner, C. W. Wester, W. H. Wilson, Roscoe Woods.

The chairman of the section, Professor E. R. Smith, presided at both the Friday afternoon and Saturday morning sessions. Dinner was enjoyed together Friday evening at Bartlett Hall. At the business meeting the following were elected officers for 1925–1926: Chairman, I. S. CONDIT, Iowa State Teachers College; Vice-Chairman, MARIAN E. DANIELLS, Iowa State College; Secretary-Treasurer, J. F. REILLY, University of Iowa.

The next regular meeting will be held at Cedar Rapids, April, 1926.

The following papers were presented:

(1) "Experiments with small numbers of observations" by Professor G. W. SNEDECOR, Iowa State College.

(2) "On the numerical effect of certain variations in the assumptions relating to the distribution of unknown probabilities in the inversion of the Bernoulli Theorem" by Professor H. L. RIETZ, University of Iowa.

(3) "Notes on algebraic deficiencies" by Professor J. V. MCKELVEY, Iowa State College.

(4) "The members of the system of conics passing through four real points" by Mr. H. K. HUGHES, University of Iowa.

(5) "The locus of the centers of the conics through four real points" by Mr. H. K. HUGHES.

(6) "An application of Rolle's theorem to functions of finite genus" by Professor MARIAN E. DANIELLS, Iowa State College.

(7) Address by the retiring chairman: "The determination of mathematical ability" by Professor E. R. SMITH, Iowa State College.

(8) "Sufficient conditions for the periodicity of the solutions of certain functional equations" by Professor W. H. WILSON, University of Iowa.

(9) "The rule of double position" by Professor J. S. TURNER, Iowa State College.

(10) "Some properties of prolate spheroids" by Professor J. S. TURNER.

(11) "On a line similar to Simson's line" by Professor ROSCOE WOODS, University of Iowa.